

Vertex Algebras and Mirror Symmetry

Lev A. Borisov

Department of Mathematics, Columbia University

2990 Broadway, Mailcode 4432, New York, NY 10027, USA

e-mail: lborisov@math.columbia.edu

Abstract

Mirror Symmetry for Calabi-Yau hypersurfaces in toric varieties is by now well established. However, previous approaches to it did not uncover the underlying reason for mirror varieties to be mirror. We are able to calculate explicitly vertex algebras that correspond to holomorphic parts of A and B models of Calabi-Yau hypersurfaces and complete intersections in toric varieties. We establish the relation between these vertex algebras for mirror Calabi-Yau manifolds. This should eventually allow us to rewrite the whole story of toric Mirror Symmetry in the language of sheaves of vertex algebras. Our approach is purely algebraic and involves simple techniques from toric geometry and homological algebra, as well as some basic results of the theory of vertex algebras. Ideas of this paper may also be useful in other problems related to maps from curves to algebraic varieties.

This paper could also be of interest to physicists, because it contains explicit description of holomorphic parts of A and B models of Calabi-Yau hypersurfaces and complete intersections in terms of free bosons and fermions.

1 Introduction

First example of Mirror Symmetry was discovered by physicists in [7]. It relates A model on one Calabi-Yau variety with B model on another one. Unfortunately, the definition of A and B models was given by physicists in terms of integrals over the set of all maps from Riemann surfaces to a given Calabi-Yau variety, see [22]. While physicists have developed good intuitive understanding of the behavior of these integrals, they are ill-defined mathematically. Nevertheless, physicists came up with predictions of numbers of rational curves of given degree in Calabi-Yau manifolds, a quintic threefold being most prominent example.

Kontsevich has introduced spaces of stable maps (see [14]) which allowed him to define mathematically virtual numbers of rational curves on a quintic. Givental proved in [10] that these virtual numbers agree with physical predictions. Because of its hard calculations, Givental's paper is a source of controversy, see [16] by Lian, Liu and Yau. In some sense, however, Givental's approach does not clarify the

origins of mirror symmetry. His proof is a beautiful and tricky calculation which has little to do with mirror involution.

The goal of this paper is to present a completely different approach to toric Mirror Symmetry which should eventually lead to conceptual understanding of mirror involution in purely mathematical terms. To do this, one has to employ the theory of vertex algebras, which is a very well developed purely algebraic theory. Malikov, Schechtman and Vaintrob have recently suggested an algebraic approach to A models (see [17]) that involves chiral de Rham complex which is a certain sheaf of vertex algebras. In my personal opinion, their paper is one of the most important mathematics papers on mirror symmetry written to date, even though it does not deal directly with mirror symmetry.

In this paper we are able to calculate A and B model vertex algebras for mirror families of Calabi-Yau complete intersections. We define (quasi-)loop-coherent sheaves over any algebraic variety X , and we show that if their sections on any affine open subset have vertex algebra structure, then the cohomology of the sheaf has this structure as well. It is hoped that the techniques of this paper will prove to be more important than the paper itself, after all, they should allow mathematicians to do rigorously what physicists have been doing half-rigorously for quite a while and with a lot of success.

As far as applications to conformal field theory are concerned, this paper suggests a way of defining A and B models for varieties with Gorenstein toroidal singularities that does not use any resolutions of such singularities. Also, we describe explicitly vertex algebras that correspond to points in fully enlarged Kähler cone (see [11]) and do not come from Calabi-Yau varieties.

The paper is organized as follows. Section 2 is devoted to (quasi-)loop-coherent sheaves, which is a generalization of the notion of (quasi-)coherent sheaves. It serves as a useful framework for the whole paper, and perhaps may have other applications. Sections 3 and 4 contain mostly background material. The only apparently new result there is Proposition 3.7, which had actually been suggested in [17]. Section 5 contains a calculation of chiral de Rham complex of a hypersurface in a smooth variety in terms of chiral de Rham complex of the corresponding line bundle. Section 6 contains a calculation which is in a sense a mirror image of the calculation of Section 5. Sections 7 and 8 combine results of previous sections to describe A and B models of Calabi-Yau hypersurfaces in smooth nef-Fano toric varieties. Sections 9 and 10 attempt to extend these results to singular varieties and complete intersections. While good progress is made there, a few details need to be further clarified. Section 11 is largely a speculation. We state there some open questions related to our construction, as well as some possible applications of the results and techniques of this paper.

We use the book of Kac [13] as the standard reference for vertex algebras.

Acknowledgments. This project began in 1995 while I was a Sloan predoc-toral fellow at the University of Michigan. I thank Martin Halpern and Christoph Schweigert who helped me learn the basics of conformal field theory. Konstantin Styrkas has answered a couple of my questions regarding vertex algebras, which has greatly improved my understanding of the subject.

2 Loop-coherent sheaves

The goal of this section is to develop foundations of the theory of (quasi-)loop-coherent sheaves over algebraic varieties. These are rather peculiar objects that nevertheless behave very much like usual coherent and quasi-coherent sheaves. For simplicity, we only concern ourselves with algebras over complex numbers. This is mostly just a formalization of the localization calculation of [17] but it provides us with a nice framework for our discussion. The idea is somehow to work with sheaves over the loop space of an algebraic variety without worrying much about infinities. Only future can tell if this is a truly useful concept or just an annoying technicality.

Definition 2.1 Let R be a commutative algebra over \mathbf{C} with a unit. R -loop-module is a vector space V over \mathbf{C} together with the following set of data. First of all, V is graded

$$V = \bigoplus_{l \geq 0} V_l.$$

We assume that values of l are integer, although it takes little effort to modify our definitions to allow any real l . We denote by $L[0]$ the grading operator, that is $L[0]v = kv$ for all $v \in V_k$. In addition, for every element $r \in R$ and every integer l there given a linear operator $r[l] : V \rightarrow V$ such that the following conditions hold

- (a) $1[k] = \delta_k^0$;
- (b) all $r[l]$ commute with each other;
- (c) $r[l]V_k \subseteq V_{k-l}$;
- (d) for every two ring elements r_1, r_2 there holds

$$\left(\sum_k r_1[k]z^{-k} \right) \left(\sum_l r_2[l]z^{-l} \right) = \sum_k (r_1 r_2)[k]z^{-k}.$$

This equation makes sense because at any given power of z while applied to any given element, only the finite number of terms on the left hand side are non-zero. This follows from (b), (c), and $L[0] \geq 0$.

Remark 2.2 R -loop-modules are usually not R -modules. Really, one has

$$(r_1 r_2)[0] = r_1[0]r_2[0] + \sum_{k \neq 0} r_1[k]r_2[-k]$$

as opposed to just $(r_1 r_2)[0] = r_1[0]r_2[0]$. However, the extra term is locally nilpotent, that is a sufficient power of it annihilates V_l for any given l . This is what makes it possible to localize loop-modules analogously to localization of usual modules.

The following proposition will also serve as a definition.

Proposition 2.3 Let S be a multiplicative system in R . Given a loop-module V over R denote by V_S its localization by the multiplicative system S_{loop} generated by $s[0]$ for all $s \in S$. We claim that V_S has a natural structure of R_S -loop-module. Moreover, $\rho : V \rightarrow V_S$ is a universal morphism in the sense that for any map to R_S -loop-module $\rho_1 : V \rightarrow V_1$ that is compatible with $R \rightarrow R_S$ there exists a unique R_S -loop-module map $\rho_2 : V_S \rightarrow V_1$ such that $\rho_1 = \rho_2 \circ \rho$.

Proof. First of all, let us provide V_S with R_S -loop-module structure. Grading is clearly unaffected by localization. For every $v \in V_S$ we have $v = w / \prod s_i[0]$ and every $a \in R_S$ looks like $a = b / \prod s_j$. We define

$$\begin{aligned} a(z)v &= \frac{1}{\prod s_i[0]} a(z)w \\ a(z)w &= b(z) \left(\prod \frac{1}{s_j}(z) \right) w \\ \frac{1}{s_j}(z)w &= \frac{1}{s_j[0] + \sum_{k \neq 0} s_j[k]z^{-k}} w = \sum_{l=0}^{\infty} (-1)^l s_j[0]^{-l-1} \left(\sum_{k \neq 0} s_j[k]z^{-k} \right)^l w \end{aligned}$$

which gives an element in $V_S[z, z^{-1}]$. Even though it seems that infinite sums appear when one applies $(1/s_j(z))$ several times, for any given w most $s_j[k]$ with positive k could be safely ignored, since they annihilate w and could always be pushed through other $s_*[*]$ by commutativity.

One, of course, needs to check that the above definition is self-consistent. It is clear that if you change v to $s[0]v/s[0]$, the result stays the same. Changing a to sa/s requires a certain calculation, but does not present any major difficulties either.

For every map $V \rightarrow V_1$ where V_1 is R_S -loop-module notice that $s[0]$ is invertible on V_1 for all $s \in S$. Really,

$$s[0](s^{-1})[0] = \mathbf{1} + \text{locally nilpotent}$$

so $s[0]$ is invertible by the same trick. Therefore, the map $V \rightarrow V_1$ can be naturally pushed through V_S . \square

Remark 2.4 Because of Remark 2.2, it is enough to localize by $s[0]$ for those s that generate S .

Proposition 2.3 allows us to define a *quasi-loop-coherent sheaf* on any complex algebraic variety X as follows.

Definition 2.5 A sheaf \mathcal{V} of vector spaces over \mathbf{C} is called *quasi-loop-coherent* if for every affine subset $\text{Spec}(R) \subset X$ sections $\Gamma(\text{Spec}(R), \mathcal{V})$ form an R -loop-module and restriction maps are precisely localization maps of Proposition 2.3.

Many results about quasi-loop-coherent sheaves could be deduced from the standard results about quasi-coherent sheaves due to the following proposition.

Proposition 2.6 For every R -loop-module V consider the following filtration

$$F^l V = \sum_{i, s_1, \dots, s_i, k_1, \dots, k_i} \prod_i s_i[k_i] V_{\leq l}.$$

We have $F^0 V \subseteq F^1 V \subseteq \dots$, and $F^{l+1} V / F^l V$ has a natural structure of R -module. Moreover, this filtration commutes with localizations.

Proof. Locally nilpotent operators $(s_1 s_2)[0] - s_1[0] s_2[0]$ push $F^{l+1}V$ to F^lV , which provides the quotient with the structure of R -module. Filtration commutes with localization, because $s[0]$ commute with $L[0]$. \square

As a result of this proposition every quasi-loop-coherent sheaf \mathcal{V} is filtered by other quasi-loop-coherent sheaves $F^l\mathcal{V}$ and all quotients are quasi-coherent. It is also worth mentioning that the above filtration is finite on every \mathcal{V}_k which prompts the following definition.

Definition 2.7 A quasi-loop-coherent sheaf is called loop-coherent, or *loco* if quasi-coherent sheaves $F^{l+1}\mathcal{V} \cap \mathcal{V}_k / F^l\mathcal{V} \cap \mathcal{V}_k$ are coherent for all k and l .

From now on we also use abbreviation *quasi-loco* in place of quasi-loop-coherent.

Remark 2.8 A zero component \mathcal{V}_0 of a (quasi-)loco sheaf is (quasi-)coherent.

Proposition 2.9 For any affine variety X and quasi-loco sheaf \mathcal{V} on it cohomology spaces $H^i(X, \mathcal{V})$ are zero for $i \geq 1$. For any projective variety X all cohomology groups of a loco sheaf are finite dimensional for each eigen-value of $L[0]$.

Proof. For both statements, one considers a specific eigen-value k of $L[0]$ and then applies an induction on l in $F^l\mathcal{V} \cap \mathcal{V}_k$. \square

Remark 2.10 There is a one-to-one functorial correspondence between R -loop-modules and quasi-loco sheaves over $\text{Spec}R$.

Remark 2.11 One can modify the definition of quasi-loco sheaves to allow negative eigen-values of $L[0]$, as long as there is some bound $L[0] \geq -N$ on them.

3 Sheaves of vertex algebras

We follow [13] in our definition of a vertex algebra. We only consider vertex algebras over \mathbf{C} .

Definition 3.1 ([13]) A vertex algebra V is first of all a super vector space over \mathbf{C} , that is $V = V_0 \oplus V_1$ where elements of V_0 are called bosonic or even and elements from V_1 are called fermionic or odd. In addition, there given a bosonic vector $|0\rangle$ called *vacuum vector*. The last part of the data that defines a vertex algebra is the so-called state-field correspondence which is a parity preserving linear map from V to $\text{End}V[[z, z^{-1}]]$

$$a \mapsto Y(a, z) = \sum_{n \in \mathbf{Z}} a_{(n)} z^{-n-1}$$

that satisfies the following axioms

- **translation covariance:** $\{T, Y(a, z)\}_- = \partial_z Y(a, z)$ where $\{\cdot, \cdot\}_-$ denotes the usual commutator and T is defined by $T(a) = a_{(-2)}|0\rangle$;
- **vacuum:** $Y(|0\rangle, z) = \mathbf{1}_V$, $Y(a, z)|0\rangle|_{z=0} = a$;
- **locality:** $(z - w)^N \{Y(a, z), Y(b, w)\}_\mp = 0$ for all sufficiently big N , where \mp is $+$ if and only if both a and b are fermionic. The equality is understood as an identity of formal power series in z and w .

We will often write $a(z)$ instead of $Y(a, z)$. Linear operators $a_{(k)}$ will be referred to as *modes* of a . Every two fields $a(z), b(w)$ of a vertex algebra have *operator product expansion*

$$a(z)b(w) = \sum_{i=1}^N \frac{c^i(w)}{(z-w)^i} + :a(z)b(w):$$

where the meaning of the symbols $\frac{1}{(z-w)^i}$ and $::$ in the above formulas is clarified in Chapter 2 of [13]. We only remark here that operator product expansion contains information about all super-commutators of the modes of a and b , and the sum on the right hand side is finite due to the locality axiom. The sum is called the singular part and the $::$ term is called the regular part. When there is no singular part, the OPE is called *non-singular* and it means that all modes of the two fields in question super-commute.

In this paper we will only use graded vertex algebras.

Definition 3.2 ([13]) A vertex algebra V is called *graded* if there is given an even diagonalizable operator H on V such that

$$\{H, Y(a, z)\}_- = z\partial_z Y(a, z) + Y(Ha, z).$$

When $Ha = ha$ we rewrite $Y(a, z)$ in the form

$$Y(a, z) = \sum_{n \in -h + \mathbb{Z}} a[n] z^{-n-h}$$

and again call $a[n]$ modes of a . Number h is usually called the *conformal weight* of a or $a(z)$. We use brackets to denote modes, which differs from the notations of [13] and [17]. Modes can hardly be confused with commutators, but we use $\{\}, \{\}_\pm$ notation for the latter just in case.

Definition 3.3 ([13]) A graded vertex algebra V is called *conformal* if there chosen an even vector v such that the corresponding field $Y(a, z) = L(z) = \sum_n L[n] z^{-n-2}$ satisfies

$$L(z)L(w) = \frac{(c/2)}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial L(w)}{z-w} + \text{reg.}$$

We also require $L[-1] = T$ and $L[0] = H$. Number c here is called *central charge* or rank of the algebra.

We now combine the theories of quasi-loco sheaves and vertex algebras to define sheaves of graded vertex algebras over an algebraic variety X . Abusing the notations, we denote the grading operator H by $L[0]$ even if algebra V is not conformal.

Definition 3.4 Let R be a commutative algebra over \mathbf{C} . A graded vertex algebra V is called vertex R -algebra if R is mapped to $L[0] = 0$ component of V so that images of all $r \in R$ are bosonic, all modes $r[n]$ commute with each other and

$$Y(r_1, z)Y(r_2, z) = Y(r_1 r_2, z).$$

In addition, we assume that $L[0]$ has only non-negative eigen-values.

Definition of graded algebra implies that $r[n]$ decreases eigen-values of $L[0]$ by n . Thus any vertex R -algebra has a structure of R -loop-module.

Proposition 3.5 Let S be a multiplicative system in R and let V be a vertex R -algebra. Then the localization V_S defined in Proposition 2.3 has a natural structure of vertex R_S -algebra.

Proof. For any element $a \in V$ and any set of elements $s_i \in S$ we need to define the field $Y(a/\prod s_i[0], z)$ on V_S . First, we do it for any element of the form $|0\rangle/\prod s_i[0]$. The corresponding field is defined, of course, as $\prod_i Y(s_i, z)^{-1}$ in agreement with Proposition 2.3. Let us check vacuum axiom of vertex algebras for this field. First of all, when we apply it to the vacuum, which has $L[0]$ eigen-value zero in any graded vertex algebra, only non-positive modes of s_i survive. As a result, at $z = 0$ we obtain precisely $|0\rangle/\prod s_i[0]$. Operator T extends naturally, because it commutes with $s[0]$. The second part of the vacuum axiom also holds, because

$$\begin{aligned} \{T, \prod_i s_i^{-1}(z)\}_- &= \{T, 1/(\prod s_i)(z)\}_- = \prod s_i^{-1}(z) \{\prod s_i(z), T\}_- \prod s_i^{-1}(z) \\ &= - \prod s_i^{-2}(z) (\partial_z \prod s_i(z)) = \partial z \prod s_i^{-1}(z). \end{aligned}$$

In addition one can show that new fields are mutually local with all old fields $Y(b, z)$ and with each other.

We will now construct a field for every element in V_S . We will do it by induction on l in filtration $F^l V$ of Proposition 2.6. For an arbitrary element $a/\prod s_i[0]$ of grading h we define

$$\begin{aligned} Y(a/\prod s_i[0], z) &= :Y(|0\rangle/\prod s_i[0], z)Y(a, z): \\ &= (\sum_{k \leq 0} (|0\rangle/\prod s_i[0])[k]z^{-k}) (\sum_i a[i]z^{-i-h}) + (\sum_i a[i]z^{-i-h}) (\sum_{k \geq 1} (|0\rangle/\prod s_i[0])[k]z^{-k}). \end{aligned}$$

One can see that this expression is well defined as an element in $\text{End}V[[z, z^{-1}]]$. When we apply this field to the vacuum, second term does not contribute, and we get $(\prod s_i^{-1})[0]a$. This is *not* the same as $a/\prod s_i[0]$ but the difference lies in the deeper part of the filtration.

So we have now constructed a field for each $a \in V_S$. To show that this definition is compatible with changes from a to $s[0]a/s[0]$, notice that all constructed fields are mutually local and satisfy the second part of vacuum axiom. Then the argument of the uniqueness theorem of Section 4.4 of [13] works and allows us to conclude that for any two ways of writing an element in V_S the corresponding fields are the same. \square

Above proposition allows us to define sheaves of vertex algebras over any algebraic variety X .

Definition 3.6 A (quasi-)loco sheaf \mathcal{V} of vector spaces over \mathbf{C} is said to be a *a (quasi-)loco sheaf of vertex algebras* if for every affine subset $\text{Spec}(R) \subset X$ sections $\Gamma(\text{Spec}(R), \mathcal{V})$ form a vertex R -algebra and restriction maps are precisely localization maps of Proposition 3.5.

Our goal now is to provide $H^*(\mathcal{V})$ with a structure of vertex algebra. Certainly, operators T and $L[0]$, as well as the vacuum, behave well under localizations and are therefore globally defined. For every affine set in X and every integer n we may consider operation (n) that maps (a, b) to $a_{(n)}b$. It also commutes with localization, which gives us the map $(n) : \mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{V}$. This map induces a cup product on the cohomology of \mathcal{V} and we shall show that combining all these maps together yields a vertex algebra structure on $H^*(\mathcal{V})$.

Proposition 3.7 Cohomology of quasi-loco sheaf of vertex algebras \mathcal{V} has a natural structure of vertex algebra. Moreover, if sections of \mathcal{V} over Zariski open sets are given the structure of conformal algebras that is compatible with localization maps, then $H^*(\mathcal{V})$ has a natural conformal structure.

Proof. We will use the equivalent set of axioms of vertex algebras, see [13], Section 4.8, This set consists of the partial vacuum identity

$$Y(|0\rangle, z) = \mathbf{1}, \quad a(-1)|0\rangle = a$$

and Borcherds identity

$$\begin{aligned} \sum_{j=0}^{\infty} \binom{m}{j} (a_{(n+j)}b)_{(m+k-j)}c &= \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} a_{(m+n-j)}(b_{(k+j)}c) \\ &\quad - (-1)^{\text{parity}(a)\text{parity}(b)} \sum_{j=0}^{\infty} (-1)^{j+n} \binom{n}{j} b_{(n+k-j)}(a_{(m+j)}c) \end{aligned}$$

for all $a, b, c \in V$ and all $k, m, n \in \mathbf{Z}$. When we have a graded algebra with $L[0] \geq 0$, for any given $L[0]$ -eigen-values of a, b and c , the sums in Borcherds identity are finite. Therefore, Borcherds identity is just a collection of identities for maps (n) between components \mathcal{V}_r of \mathcal{V} . Therefore, they induce identities on cohomology of \mathcal{V} when we replace (n) with corresponding cup products. A careful examination of signs shows that Borcherds identity holds on cohomology of \mathcal{V} if we define parity on $H^s(\mathcal{V})$ as the sum of s and parity on \mathcal{V} . Partial vacuum identity on cohomology also follows from the vacuum identity on \mathcal{V} and the fact that $|0\rangle$ is globally defined. If \mathcal{V} has conformal structure, then $a = L[-2]|0\rangle$ is globally defined, so it lies in H^0 and provides cohomology with conformal structure of the same central charge. \square

We finish this section with discussion of *BRST cohomology*. Let V be a vertex algebra and let a be an element of V such that $a_{(0)}^2 = 0$. Consider cohomology of V with respect to $a_{(0)}$, which is called BRST cohomology. Operator $a_{(0)}$ and field $Y(a, z)$ are called BRST operator and BRST field respectively. The following proposition is standard.

Proposition 3.8 BRST cohomology of V with respect to $a_{(0)}$ has a natural structure of vertex algebra.

Proof. One has the following identity ([13], equation 4.6.9)

$$\{a_0, Y(b, z)\}_{\pm} = Y(a_{(0)}b, z).$$

Therefore, if b is annihilated by $a_{(0)}$ then $Y(b, z)$ commutes with $a_{(0)}$ and conserves the kernel and the image of BRST operator. This provides us with the set of mutually local fields for BRST cohomology, and it remains to employ the uniqueness theorem of Section 4.4 of [13]. \square

In particular, if a is an odd element of V such that all modes $a_{(n)}$ anticommute with each other, then $a_{(0)}$ could serve as BRST operator. All major results of this paper involve BRST cohomology by operators of this type.

4 Chiral de Rham complex as a sheaf of vertex algebras over a smooth variety

In this section we review and summarize results of the extremely important paper of Malikov, Schechtman and Vaintrob [17]. Notations of our paper follow closely those of [17]. We assume some familiarity with the vertex algebras of free bosons and fermions. The reader is referred to [13] or pretty much any conformal field theory textbook.

For every smooth variety X authors of [17] define a loco sheaf of vertex algebras $\mathcal{MSV}(X)$ which is called *chiral de Rham complex* of X . It is described in local coordinates $x^1, \dots, x^{\dim X}$ as follows. There are given $2\dim X$ fermionic fields $\varphi^i(z)$, $\psi_i(z)$ and $2\dim X$ bosonic fields $a_i(z)$, $b^i(z)$ where index i is allowed to run from 1 to $\dim X$. The non-trivial super-commutators between modes are given by

$$\begin{aligned} \{a_i[k], b^j[l]\}_- &= \delta_i^j \delta_{k+l}^0 \\ \{\psi_i[k], \varphi^j[l]\}_+ &= \delta_i^j \delta_{k+l}^0, \end{aligned}$$

and the fields are defined as

$$\begin{aligned} a_i(z) &= \sum_k a_i[k] z^{-k-1}, \quad b^i(z) = \sum_k b^i[k] z^{-k}, \\ \varphi^i(z) &= \sum_k \varphi^i[k] z^{-k}, \quad \psi_i(z) = \sum_k \psi_i[k] z^{-k-1}. \end{aligned}$$

There is defined a Fock space generated from the vacuum vector $|0\rangle$ by non-positive modes of b and φ and by negative modes of a and ψ . To obtain sections of $\mathcal{MSV}(X)$ over a neighborhood U of $x = 0$, one considers tensor product V of the above Fock space with the ring of functions over U with $b^i[0]$ plugged in place of x^i . Tensor product is taken over the ring $\mathbf{C}[b[0]]$. Grading on this space is defined as the opposite of the sum of mode numbers.

Certainly one needs to specify how elements of V change under a change of local coordinates. For each new set of coordinates

$$\tilde{x}^i = g^i(x), \quad x^j = f^j(\tilde{x}^j)$$

this is accomplished in [17] by the formulas

$$\tilde{b}^i(z) = g^i(b(z))$$

$$\begin{aligned}
\tilde{\varphi}^i(z) &= g_j^i(b(z))\varphi^j(z) \\
\tilde{a}_i(z) &= :a_j(z)f_i^j(b(z)):+:\psi_k(z)f_{i,l}^k(b(z))g_r^l(b(z))\varphi^r(z): \\
\tilde{\psi}_i(z) &= \psi_j(z)f_i^j(b(z))
\end{aligned}$$

where

$$g_j^i = \partial g^i / \partial x^j, \quad f_j^i = (\partial f^i / \partial \tilde{x}^j) \circ g, \quad f_{j,k}^i = (\partial^2 f^i / \partial \tilde{x}^j \partial \tilde{x}^k) \circ g$$

and normal ordering $::$ is defined by pushing all positive modes of a , b , ψ and φ to the right, multiplying by (-1) whenever two fermionic modes are switched.

For any choice of local coordinates, one introduces fields

$$\begin{aligned}
L(z) &= : \partial_z b^i(z) a_i(z) : + : \partial_z \varphi^i(z) \psi_i(z) : \\
J(z) &= : \varphi^i(z) \psi_i(z) : \\
G(z) &= \partial_z b^i(z) \psi_i(z) \\
Q(z) &= \partial_z a_i(z) \varphi^i(z).
\end{aligned}$$

Field $L(z)$ is invariant under the change of coordinates, which provides $\mathcal{MSV}(X)$ with structure of sheaf of conformal vertex algebras. The $L[0] = 0$ part is naturally isomorphic to the usual de Rham complex on X , with grading given by eigenvalues of $J[0]$ and differential given by $Q[0]$ (both modes are globally defined). If X is Calabi-Yau, all four of the above fields are well-defined which provides $\mathcal{MSV}(X)$ with structure of sheaf of *topological vertex algebras*. This means that spaces of sections over affine subsets are equipped with structures of topological vertex algebras, in a manner consistent with localization. This structure is analogous to the conformal structure but requires a choice of four fields Q , G , J and L that satisfy certain OPEs, see [17].

It was suggested in [17] that cohomology of $\mathcal{MSV}(X)$ has a structure of vertex algebra which describes the holomorphic part of A model of X , see [22]. Since we now know how to provide cohomology of the loco sheaf of vertex algebras $\mathcal{MSV}(X)$ with such structure, we can state the following definition.

Definition 4.1 Let X be a smooth algebraic variety over \mathbf{C} . We define *A model topological vertex algebra* of X to be $H^*(\mathcal{MSV}(X))$ with the structure of vertex algebra on it defined in Proposition 3.7. This algebra also possesses the conformal structure since $L(z)$ is globally defined, as well as the structure of the topological algebra, with operators given by formulas of [17] in the case when X is Calabi-Yau.

If X is a Calabi-Yau variety, one can also talk about B model topological vertex algebra of X . As a vertex algebra, it is identical to the A model, but the additional structure of the topological algebra differs.

Definition 4.2 Let X be a smooth Calabi-Yau manifold. We define *B model topological vertex algebra* of X as follows. As a vector space, it coincides with the A model topological vertex algebra of X . Operator T is also the same, and so are the

fields of the algebra. Topological structure of the B model vertex algebra is related to the topological structure of A model algebra by *mirror involution*

$$Q_B = G_A, \quad G_B = Q_A, \quad J_B = -J_A, \quad L_B = L_A - \partial J_A.$$

In what follows we will often abuse the notations and call these algebras simply A and B model vertex algebras, but it should always be understood that they are considered together with their extra structure. Notice that B model is ill-defined for varieties X that are not Calabi-Yau, even as a conformal vertex algebra, because L_B is ill-defined for them.

The first goal of our paper is to calculate A and B model vertex algebras for Calabi-Yau hypersurfaces in smooth toric nef-Fano varieties. The second goal (which is only partially achieved) is to generalize our results to toric varieties with Gorenstein singularities.

5 Vertex algebras of line bundles and zeros of their sections

In this section we study chiral de Rham complex of a line bundle L over a smooth variety P . Given a section of the dual line bundle we are able to calculate the chiral de Rham complex of its zero set in terms of the push-forward of the chiral de Rham complex of the line bundle.

We denote the projection to the base by $\pi : L \rightarrow P$. The line bundle structure is locally described by the fact that there is one special coordinate x^1 such that allowed local changes of coordinates are compositions of changes in $x^2, \dots, x^{\dim L}$ and changes

$$\tilde{x}^1 = x^1 h(x^2, \dots, x^{\dim L}), \quad \tilde{x}^i = x^i, \quad i \geq 2.$$

Proposition 5.1 Field $b^1(z)\psi_1(z)$ depends only on the line bundle structure of L .

Proof. This field is clearly unaffected by any changes of coordinates on the base that leave x^1 intact. In addition, for the change of coordinates as above, we have

$$\tilde{b}^1 = b^1 h(b^2, \dots, b^{\dim L}), \quad \tilde{\psi}_1 = \psi_1 / h(b^2, \dots, b^{\dim L}).$$

As a result, field $b^1(z)\psi_1(z)$ is independent from the choice of local coordinates that are compatible with the given line bundle structure. \square

The following lemma is clear.

Lemma 5.2 Line bundle L has trivial canonical class if and only if L is the canonical line bundle on P .

Remark 5.3 Since $\mathcal{MSV}(L)$ is a loco sheaf, its cohomology spaces $H^*(\mathcal{MSV}(L))$ are isomorphic to cohomology $H^*(\pi_*\mathcal{MSV}(L))$ of its push-forward to P , which is the sheaf we are mostly interested in.

Remark 5.4 One may also consider the bundle ΠL obtained by declaring the coordinate on L odd. It turns out that the corresponding sheaf of vertex algebras is roughly the same as the corresponding sheaf for the even bundle L^{-1} . More precisely, the push-forwards of both bundles to P coincide. Locally the isomorphism is obtained by mapping $(b^1, \varphi^1, a_1, \psi_1)$ for ΠL to $(\psi_1, a_1, \varphi^1, b^1)$ for L^{-1} . It has been observed in [20] that mirror symmetry for Calabi-Yau complete intersections may be formulated in terms of odd bundles on ambient projective varieties.

Let us additionally assume that we have in our disposal a section μ of the dual line bundle L^{-1} . This amounts to having a global function on L which is linear on fibers. We will also assume that zeros of μ form a reduced non-singular divisor X on P . The goal of the rest of the section is to describe $\mathcal{MSV}(X)$ in terms of $\pi_*\mathcal{MSV}(L)$.

The following lemma is easily checked by a calculation in local coordinates. In fact, it holds for any global function on any smooth variety.

Lemma 5.5 Fields $\mu(z)$ and $D\mu(z)$ that are locally defined as

$$\mu(z) = \mu(b^1, \dots, b^{\dim L})(z), \quad D\mu(z) = \sum_i \varphi^i(z) \frac{\partial \mu}{\partial b^i}(z)$$

are independent of the choice of coordinates and are therefore globally defined. In particular, the operator $\mathcal{BRST}_\mu = \oint D\mu(z) dz$ is globally defined.

It is now time to state the main result of this section.

Theorem 5.6 Let X be a smooth hypersurface in a smooth variety P defined as above by a section μ of line bundle L^{-1} . Then sheaf of vertex algebras $\mathcal{MSV}(X)$ is isomorphic to BRST cohomology of sheaf $\pi_*\mathcal{MSV}(L)$ with respect to the operator \mathcal{BRST}_μ . Here cohomology is understood in the sense of sheaves, that is as a sheafification of BRST cohomology presheaf.

Proof. Clearly, \mathcal{BRST}_μ is a differential, because its anticommutator with itself is zero. We denote the BRST cohomology of $\pi_*\mathcal{MSV}(L)$ by $\widehat{\mathcal{MSV}}(X)$ and it is our goal to construct an isomorphism between $\widehat{\mathcal{MSV}}(X)$ and $\mathcal{MSV}(X)$.

It is enough to construct this isomorphism locally for any point $p \in P$ provided that our construction withstands a change of coordinates. We use Hausdorff topology on P rather than Zariski topology. Point p may or may not lie in X so our discussion splits into two cases.

Case 1. $p \notin X$

In this case for any sufficiently small neighborhood $U \subset P$ of p we can choose a coordinate system $(x^1, x^2, \dots, x^{\dim L})$ on $\pi^{-1}U$ such that x_1 is the special line bundle variable, and $\mu = x^1$. As a result,

$$\mathcal{BRS}\mathcal{T}_\mu = \oint \varphi^1(z) dz = \varphi^1[1].$$

A simple calculation on the flat space then shows that cohomology by $\mathcal{BRS}\mathcal{T}_\mu$ are zero on sections of $\pi_*\mathcal{MSV}(L)$ for any sufficiently small U . As a result, $\mathcal{MSV}(X)$ is supported on X , which is, of course, true of $\mathcal{MSV}(X)$.

Case 2. $p \in X$.

For any sufficiently small neighborhood $U \subset P$ of p we can choose a system of coordinates $(x^1, x^2, \dots, x^{\dim L})$ on $\pi^{-1}U$ that agrees with line bundle structure, such that $\mu = x^1 x^2$ and $x^3, \dots, x^{\dim L}$ form a system of local coordinates on $X \cap U$. We then have

$$\mathcal{BRS}\mathcal{T}_\mu = \oint (b^1 \varphi^2 + b^2 \varphi^1)(z) dz = \sum_{k \in \mathbf{Z}} (b^1[k] \varphi^2[-k+1] + b^2[k] \varphi^1[-k+1]).$$

Fock space $\Gamma(U, \pi_*\mathcal{MSV}(L))$ is a tensor product of spaces $\text{Fock}_{1,2}$ and $\text{Fock}_{\geq 3}$, which are the spaces generated by modes of $a_i, b^i, \varphi^i, \psi_i$ for $i \in \{1, 2\}$ and $i \in \{3, \dots, \dim L\}$ respectively. Since $\mathcal{BRS}\mathcal{T}_\mu$ acts on the first component of this tensor product, its cohomology is isomorphic to the tensor product of $\text{Fock}_{\geq 3}$ and cohomology of $\text{Fock}_{1,2}$ with respect to $\mathcal{BRS}\mathcal{T}_\mu$.

We claim that cohomology of $\text{Fock}_{1,2}$ with respect to $\mathcal{BRS}\mathcal{T}_\mu$ is one-dimensional and is generated by the image of vacuum vector $|0\rangle$. We do not multiply by $\Gamma(U, \mathcal{O}_P)$, which does not alter the argument. Notice first that $\text{Fock}_{1,2}$ is a restricted tensor product (that is almost all factors are 1) of the following infinite set of vector spaces.

- $\bigoplus_{l \geq 0} \mathbf{C}(a^1[-k])^l \bigoplus_{l \geq 0} \mathbf{C}(a^1[-k])^l \varphi^2[-k+1]$, for all $k > 0$
- $\bigoplus_{l \geq 0} \mathbf{C}(a^2[-k])^l \bigoplus_{l \geq 0} \mathbf{C}(a^2[-k])^l \varphi^1[-k+1]$, for all $k > 0$
- $\bigoplus_{l \geq 0} \mathbf{C}(b^1[-k])^l \bigoplus_{l \geq 0} \mathbf{C}(b^1[-k])^l \psi^2[-k-1]$, for all $k \geq 0$
- $\bigoplus_{l \geq 0} \mathbf{C}(b^2[-k])^l \bigoplus_{l \geq 0} \mathbf{C}(b^2[-k])^l \psi^1[-k-1]$, for all $k > 0$
- $\bigoplus_{l \geq 0} R(b^2[-k])^l \bigoplus_{l \geq 0} R(b^2[-k])^l \psi^1[-k-1]$, for all $k > 0$

In the last formula R means the ring of function on a disc. We assume here that the neighborhood U is a product of $|x^2| \leq c$ and some $U_{x^3, \dots, x^{\dim L}}$.

Vacuum vector, of course, corresponds to the product of all 1. The Fock space is graded by the eigen-values of $L_{1,2}[0]$ that is by the opposite of the total sum of indices. Operator $\mathcal{BRS}\mathcal{T}_\mu$ shifts this grading by -1 . If we consider elements

with bounded grading, it is enough to consider only product of a finite number of above spaces. For each such product, \mathcal{BRST}_μ is a sum of anticommuting operators on each component. One can then show that cohomology is a tensor product of cohomologies for each component by induction on the number of components. On each step of the induction we are using a spectral sequence for a stupid filtration of the tensor product complex, with grading given by eigen-values of $L_{1,2}[0]$.

As a result, to show that cohomology space is one-dimensional, it is enough to show that for each of the spaces above cohomology is one-dimensional and is given by the image of 1. It is sufficient to consider the first, third, and fifth types only. For a space of first type, kernel of \mathcal{BRST}_μ is

$$\mathbf{C}1 \oplus \bigoplus_{l \geq 0} \mathbf{C}(a^1[-k])^l \varphi^2[-k+1]$$

and its image is

$$\bigoplus_{l \geq 0} \mathbf{C}(a^1[-k])^l \varphi^2[-k+1]$$

so the image of 1 generates cohomology. For a space of third type, kernel is

$$\bigoplus_{l \geq 0} \mathbf{C}(b^1[-k])^l$$

and image is

$$\bigoplus_{l \geq 1} \mathbf{C}(b^1[-k])^l$$

which gives the same result. For the space of the fifth type, we use $R/xR = \mathbf{C}$.

So we managed to show that for a given choice of coordinates on $\pi^{-1}U$, there is an isomorphism between sections of $\widehat{\mathcal{MSV}}(X)$ and $\mathcal{MSV}(X)$. The proof is not over yet, because we need to show that these locally defined isomorphisms could be glued together. This amounts to the demonstration that the isomorphism just constructed commutes with any changes of coordinates on $\pi^{-1}U$ that preserve our setup. Every such coordinate change could be written in the form

$$\tilde{x}^1 = x^1 \cdot h(x^2, \dots, x^{\dim L}), \quad \tilde{x}^2 = x^1/h(x^2, \dots, x^{\dim L})$$

$$\tilde{x}^i = f^i(x^3, \dots, x^{\dim L}) + x^2 g^i(x^2, \dots, x^{\dim L}), \quad i \geq 3.$$

It is clear that when $h = 1$ and $g^i = 0$ the corresponding splitting of the Fock space is unaffected and the resulting isomorphism precisely matches the change of variables on X . As a result, we only need to show that isomorphism commutes with coordinate changes such that $f^i(x) = x^i$.

One can show that in this case fields $\tilde{a}_i(z), \tilde{b}^i(z), \tilde{\varphi}^i(z), \tilde{\psi}_i(z)$ for $i \geq 3$ act on the cohomology in the same way as the operators $a_i(z), b^i(z), \varphi^i(z), \psi_i(z)$, because the difference lies in the image of \mathcal{BRST}_μ . This finishes the proof. \square

It is clear that our isomorphism commutes with structures of sheaves of vertex algebras. We also have the following corollary which will be very useful later.

Proposition 5.7 For any affine subset $U \subset P$ the \mathcal{BRST}_μ cohomology space of $\Gamma(U, \pi_* \mathcal{MSV}(L))$ is isomorphic to $\Gamma(U, \mathcal{MSV}(X))$.

Proof. Sheaf $\pi_*\mathcal{MSV}(L)$ is a quasi-loco sheaf, and \mathcal{BRST}_μ is a map of a quasi-loco sheaf into itself. For any affine subset it is induced from the map of corresponding loop-modules, and then everything follows from the Remark 2.10. \square

We are especially interested in the case where L has a non-degenerate top form. In this case, by Lemma 5.2, L is a canonical line bundle, and a section μ of L^{-1} produces a Calabi-Yau divisor X on P . Our goal here is to calculate global fields $G(z)$ and $Q(z)$ on $\mathcal{MSV}(X)$ in terms of some global fields on $\pi_*\mathcal{MSV}(L)$.

Proposition 5.8 When L is canonical bundle on X , field $G_X(z)$ is the image of field $G_L(z) - (b_1(z)\psi_1(z))'$. Field $Q_X(z)$ is the image of field $Q_L(z)$.

Proof. Because of 5.1, all fields in question are defined globally, so a local calculation is sufficient. We assume notations of the proof of Theorem 5.6. Then we have

$$\begin{aligned} Q_X(z) - Q_L(z) &= -a_1(z)\varphi^1(z) - a_2(z)\varphi^2(z), \\ G_X(z) - G_L(z) + (b^1(z)\psi_1(z))' &= b^2(z)'\psi_2(z) - b^1(z)\psi_1(z)', \end{aligned}$$

and we need to show that right-hand sides of these equations are commutators of \mathcal{BRST}_μ and some fields. This goal is accomplished by fields $-a_1(z)a_2(z)$ and $-\psi_1'(z)\psi_2(z)$ respectively. \square

6 BRST description of vertex algebra in logarithmic coordinates

This section is in a sense a mirror of the previous one. It contains a local calculation of chiral de Rham complex of a smooth toric variety as BRST cohomology of some \mathcal{MSV} -like space defined in terms of local coordinates.

We introduce some notations which will stay with us for the rest of the paper. Let M be a free abelian group of rank $\dim M$ and $N = \text{Hom}(M, \mathbf{Z})$ be its dual. Vector space $(M \oplus N) \otimes \mathbf{C}$ has dimension $2\dim M$ and it is equipped with a standard bilinear form denoted by \cdot, \cdot . This allows us to construct $2\dim M$ bosonic and $2\dim M$ fermionic fields. Really, one can always construct k bosonic and k fermionic fields starting from a vector space of dimension k with a non-degenerate bilinear form on it, see for example [13], so our purpose here is to fix notations. For every $m \in M$ and $n \in N$ we have

$$\begin{aligned} m \cdot B(z) &= \sum_{k \in \mathbf{Z}} m \cdot B[k] z^{-k-1}, \quad n \cdot A(z) = \sum_{k \in \mathbf{Z}} n \cdot A[k] z^{-k-1}, \\ m \cdot \Phi(z) &= \sum_{k \in \mathbf{Z}} m \cdot \Phi[k] z^{-k}, \quad n \cdot \Psi(z) = \sum_{k \in \mathbf{Z}} n \cdot \Psi[k] z^{-k-1}. \end{aligned}$$

Notice that the moding of B also has z^{-k-1} in it, in contrast to the moding of b^i in the previous section. The non-zero super-commutators are

$$\{m \cdot B[k], n \cdot A[l]\}_- = (m \cdot n)k\delta_{k+l}^0 \mathbf{id},$$

$$\{m \cdot \Phi[k], n \cdot \Psi[l]\}_+ = (m \cdot n) \delta_{k+l}^0 \mathbf{id}.$$

Our battlefield will be the following space whose construction is standard as well.

$$\begin{aligned} \text{Fock}_{M \oplus N} =_{\text{def}} & \bigoplus_{m \in M, n \in N} \otimes_{k \geq 1} \mathbf{C}[B[-k]] \otimes_{k \geq 1} \mathbf{C}[A[-k]] \otimes_{l \geq 0} (\mathbf{C} + \mathbf{C}\Phi[-l]) \\ & \otimes_{l \geq 1} (\mathbf{C} + \mathbf{C}\Psi[-l]) |m, n\rangle \end{aligned}$$

Here \otimes means restricted tensor product over \mathbf{C} where only finitely many factors are not equal to 1. Vectors $|m, n\rangle$ are annihilated by positive modes of A , B , and Φ , and by non-negative modes of Ψ . Also,

$$A[0]|m, n\rangle = m|m, n\rangle, \quad B[0]|m, n\rangle = n|m, n\rangle.$$

This Fock space possesses a structure of vertex algebra, see for example [13]. Among the fields of this algebra the important role is played by so-called *vertex operators*

$$: e^{\int (m \cdot B(z) + n \cdot A(z))} :$$

which are defined as follows

$$\begin{aligned} & : e^{\int (m \cdot B(z) + n \cdot A(z))} : \prod A[\dots] \prod B[\dots] \prod \Phi[\dots] \prod \Psi[\dots] |m_1, n_1\rangle \\ & = C(m, n, m_1, n_1) z^{m \cdot n_1 + n \cdot m_1} \prod_{n < 0} e^{-(m \cdot B[n] + n \cdot A[n]) \frac{z^n}{n}} \prod_{n > 0} e^{-(m \cdot B[n] + n \cdot A[n]) \frac{z^n}{n}} \prod A[\dots] \\ & \quad \prod B[\dots] \prod \Phi[\dots] \prod \Psi[\dots] |m + m_1, n + n_1\rangle. \end{aligned}$$

Cocycle $C(m, n, m_1, n_1)$ here equals $(-1)^{m \cdot n_1}$. It is used to make vertex operators purely bosonic. Our notation suppresses this cocycle, which is a bit unusual but should not lead to any confusion. Vertex operators obey the following OPEs

$$: e^{\int (m \cdot B(z) + n \cdot A(z))} :: e^{\int (m_1 \cdot B(w) + n_1 \cdot A(w))} := \frac{: e^{\int (m \cdot B(z) + n \cdot A(z))} e^{\int (m_1 \cdot B(w) + n_1 \cdot A(w))} :}{(z - w)^{m \cdot n_1 + n \cdot m_1}}$$

where putting both fields under the same $::$ sign means that we move all negative modes to the left and all positive modes to the right as in the definition of vertex operators above. Of course, this OPE could be expanded by Taylor formula, and the resulting fields are normal ordered products of vertex operators, free bosons, and their derivatives. In general, all fields of the vertex algebra $\text{Fock}_{M \oplus N}$ are normal ordered products of various B , A , Ψ , Φ and their derivatives times one (perhaps trivial) vertex operator.

Remark 6.1 Vertex algebra $\text{Fock}_{M \oplus N}$ possesses a conformal structure, given by

$$L_{M \oplus N}(z) = : B(z) \cdot A(z) : + : (\partial_z \Phi(z)) \cdot \Psi(z) :.$$

The corresponding grading operator $L_{M \oplus N}[0]$ assigns grading $m \cdot n$ to vector $|m, n\rangle$. Elements $\prod_i m_i \cdot \Phi[0] |m, n\rangle$ have the same eigenvalue, but for every other element with the same $A[0]$ and $B[0]$ eigen-values, the grading is strictly larger than $m \cdot n$. Under this conformal structure the grading of the fields A , B , Ψ and Φ is as above.

The goal of this section is to construct a flat space vertex algebra in terms of A, B, Φ, Ψ . We first look at the case of dimension one. In this case M and N are one-dimensional, and A, B, Φ, Ψ are no longer vector-valued. Consider the following fields

$$b(z) = e^{\int B(z)}, \quad \varphi(z) = \Phi(z)e^{\int B(z)}, \quad \psi(z) = \Psi(z)e^{-\int B(z)},$$

$$a(z) = :A(z)e^{-\int B(z)}: + : \Phi(z)\Psi(z)e^{-\int B(z)}:.$$

Proposition 6.2 Operator product expansions of a, b, φ, ψ are

$$a(z)b(w) = \frac{1}{z-w} + \text{reg.}, \quad \varphi(z)\psi(w) = \frac{1}{z-w} + \text{reg.},$$

and all other OPEs are non-singular.

Proof. It is a standard calculation of OPEs that include vertex operators, which we omit. This is not too surprising since the fields in question are given by the formulas of [17] applied to the exponential change of variables $\tilde{x} = \exp(x)$. \square

Proposition 6.3 Modes of the fields a, b, φ, ψ generate a vertex algebra which is isomorphic to global sections of chiral de Rham complex of a one-dimensional affine space.

Proof. First, all OPEs are correct due to 6.2. Since a and b are bosonic and φ and ψ are fermionic, this implies that super-commutators of their modes are correct. Notice that the conformal weight of b is zero, so it is moded correctly. One can also show that positive modes of b, φ and non-negative modes of a, ψ annihilate $|0, 0\rangle$. The rest follows from the fact that Fock representation of the algebra of modes is irreducible. \square

The following calculation is extremely useful.

Proposition 6.4 We define $L(z), J(z), Q(z)$, and $G(z)$ for a, b, φ, ψ as usual, see [17]. Then in terms of A, B, Φ, Ψ we have

$$Q(z) = A(z)\Phi(z) - \partial_z\Phi(z), \quad G(z) = B(z)\Psi(z),$$

$$J(z) = : \Phi(z)\Psi(z) : + B(z)$$

$$L(z) = : B(z)A(z) : + : \partial_z\Phi(z)\Psi(z) :.$$

Proof. It is a standard calculation, which is again omitted. \square

We now define $\text{Fock}_{M \oplus N_{\geq 0}}$ as a subalgebra of $\text{Fock}_{M \oplus N}$ characterized by the condition that eigen-values of $B[0]$ are non-negative. This amounts to only allowing $|m, n\rangle$ with $n \geq 0$. We will now show that vertex algebra generated by a, b, φ, ψ could be obtained as a certain BRST cohomology of vertex algebra $\text{Fock}_{M \oplus N_{\geq 0}}$.

Theorem 6.5 Vertex algebra of a, b, φ, ψ is isomorphic to BRST cohomology of $\text{Fock}_{M \oplus N \geq 0}$ with respect to the operator

$$\mathcal{BRST}_g = \oint \mathcal{BRST}_g(z) dz = \oint g \Psi(z) e^{\oint A(z)} dz$$

where g is an arbitrary non-zero complex number.

Proof. First of all, notice that all modes of a, b, φ, ψ commute with \mathcal{BRST}_g . Really, all these fields except $a(z)$ give non-singular OPEs with $\mathcal{BRST}_g(w)$, and

$$\begin{aligned} a(z)\mathcal{BRST}_g(w) &= \frac{g : A(z)\Psi(w) e^{\int -B(z)+A(w)} :}{z-w} + \text{reg.} \\ &+ g\left(-\frac{\Psi(z)}{z-w} + O(z-w)\right) \frac{e^{-\int B(z)+\int A(w)}}{z-w} \\ &= \frac{-g\Psi(z) : e^{\int (A(z)-B(z))} :}{(z-w)^2} + \text{reg.} \end{aligned}$$

which implies $\{a(z), \mathcal{BRST}_g\}_- = 0$.

Space $\text{Fock}_{M \oplus N \geq 0}$ is graded by eigen-values of $B[0]$ and \mathcal{BRST}_g shifts them by one. We first show that \mathcal{BRST}_g has no cohomology for eigen-values of $B[0]$ that are positive. Really, we can look at the operator $\mathcal{R}(z) = \Phi(z) e^{-\int A(z)}$. A similar calculation shows that

$$\{\mathcal{R}(z), \mathcal{BRST}_g\}_+ = \mathbf{id}$$

and therefore the anticommutator of the zeroth mode of $R(z)$ and \mathcal{BRST}_g is identity. Thus we found a homotopy operator, which insures that there is no cohomology at positive eigen-values of $B[0]$.

Fortunately, the above operator shoots out of $\text{Fock}_{M \oplus N \geq 0}$ from zero eigen-space of $B[0]$. So we found that the cohomology is isomorphic to the kernel of \mathcal{BRST}_g on the zero eigen-space of $B[0]$. To show that all elements of this space can be obtained by applying modes of a, b, φ, ψ to $|0, 0\rangle$, we employ the result of Proposition 6.4. More precisely, $L[0]$ has non-negative eigen-values. Moreover, its zero eigen-space is

$$\bigoplus_{m \in \mathbf{Z}} (\mathbf{C} \oplus \mathbf{C}\Phi[0]) |m, 0\rangle .$$

Since $L[0]$ commutes with \mathcal{BRST}_g , it induces grading on the kernel.

We prove by induction on eigen-values of $L[0]$ that all elements of the kernel of \mathcal{BRST}_g with zero eigen-value of $B[0]$ are obtained by applying modes of a, b, φ, ψ to $|0, 0\rangle$. For $L[0] = 0$ notice that cohomology is graded by eigen-values of $A[0]$. An explicit calculation then shows that for $k < 0$ elements $\mathcal{BRST}_g|k, 0\rangle$ and $\mathcal{BRST}_g\Phi[0]|k, 0\rangle$ are linearly independent. In addition, $\mathcal{BRST}_g\Phi[0]|0, 0\rangle$ is non-zero (it is proportional to $|0, 1\rangle$), and the rest is generated by modes of b and φ .

If $L[0]v = lv$ with $l > 0$, notice that

$$L[0] = \sum_{k < 0} ka[k]b[-k] + \sum_{k > 0} kb[-k]a[k] - \sum_{k < 0} k\psi[k]\varphi[-k] + \sum_{k > 0} k\varphi[-k]\psi[k].$$

When applied to v , only finitely many terms survive. So we have

$$v = \frac{1}{l} \sum_i p_i q_i v.$$

Since v is in the kernel of \mathcal{BRST}_g , and p_i commutes with \mathcal{BRST}_g , $p_i v$ is in the kernel for each i . Also $p_i v$ has strictly lower eigen-value of $L[0]$, so it is generated by modes of a, b, φ, ψ due to the induction assumption. Therefore, v is also generated by modes of a, b, φ, ψ , which finishes the proof. \square

We can extend this theorem to lattices of any dimension as follows. Consider a primitive cone K^* in lattice N . Primitive here means that it is generated by a basis $n_1, \dots, n_{\dim N}$ of N . The dual basis is denoted by $m_1, \dots, m_{\dim M}$. We denote by $\text{Fock}_{M \oplus K^*}$ the subalgebra of $\text{Fock}_{M \oplus N}$ where eigen-values of $B[0]$ are allowed to lie in K^* . We consider vertex algebra of flat space that is generated by fields

$$b^i(z) = e^{\int m_i \cdot B(z)}, \quad \varphi^i(z) = (m_i \cdot \Phi(z)) e^{\int m_i \cdot B(z)}, \quad \psi_i(z) = (n_i \cdot \Psi(z)) e^{-\int m_i \cdot B(z)},$$

$$a_i(z) = : (n_i \cdot A(z)) e^{-\int m_i \cdot B(z)} : + : (m_i \cdot \Phi(z)) (n_i \cdot \Psi(z)) e^{-\int m_i \cdot B(z)} :$$

for all $i = 1, \dots, \dim M$.

Theorem 6.6 Vertex algebra of $a_i, b^i, \varphi^i, \psi_i$ is isomorphic to BRST cohomology of $\text{Fock}_{M \oplus K^*}$ with respect to operator

$$\mathcal{BRST}_g = \oint \mathcal{BRST}(z) dz = \oint \sum_i g_i (n_i \cdot \Psi(z)) e^{\oint n_i \cdot A(z)} dz$$

where $g_1, \dots, g_{\dim M}$ are arbitrary non-zero complex numbers. Moreover, operators L, J, G and Q are given by

$$Q(z) = A(z) \cdot \Phi(z) - \deg \cdot \partial_z \Phi(z), \quad G(z) = B(z) \cdot \Psi(z),$$

$$J(z) = : \Phi(z) \cdot \Psi(z) : + \deg \cdot B(z)$$

$$L(z) = : B(z) \cdot A(z) : + : \partial_z \Phi(z) \cdot \Psi(z) :$$

where "deg" is an element in M that equals 1 on all generators of K^* .

Proof. The result follows immediately from Theorem 6.5. Really, $\text{Fock}_{M \oplus K^*}$ is a tensor product of $\dim M$ spaces discussed in there. We grade each space by eigen-values of $m_i \cdot B[0]$, and \mathcal{BRST}_{g_i} becomes a degree one differential. Operator \mathcal{BRST}_g is a total differential on the corresponding total complex, which finishes the proof. Another option is to go through the proof of Theorem 6.5 with minor changes due to higher dimension. \square

7 Smooth toric varieties and hypersurfaces

The result of the previous section could be interpreted as a calculation of chiral de Rham complex for a smooth affine toric variety given by cone $K^* \subset N$. The first objective of this section is to learn how to glue these objects together to get chiral de Rham complex of a smooth toric variety (or rather a canonical line bundle over it). Then we employ the result of Theorem 5.6 to calculate vertex algebras of Calabi-Yau hypersurfaces in toric varieties.

Let us recall the set of data that defines a smooth toric variety. For general theory of toric varieties see [8, 9, 19]. Paper of Batyrev [1] may also be helpful. A toric variety \mathbf{P}_Σ is given by *fan* Σ which is a collection of rational polyhedral cones in N with vertex at 0 such that

- (a) for any cone C^* of Σ , $C^* \cap (-C^*) = \{0\}$;
- (b) if two cones intersect, their intersection is a face in both of them;
- (c) if a cone lies in Σ , then all its faces lie in Σ , this also includes the vertex (zero-dimensional face) of the cone.

Toric variety is smooth if and only if all cones in Σ are basic, that is generated by a part of a basis of N .

For every cone $C^* \in \Sigma$, one considers the dual cone $C \in M$ defined by

$$C = \{m \in M, \text{ s.t. } m \cdot C^* \geq 0\}$$

and the corresponding affine variety $\mathbf{A}_C = \text{Spec}(\mathbf{C}[C])$. We employ a multiplicative notation and denote elements of $\mathbf{C}[M]$ by x^m for all $m \in M$. If C_1^* is a face of C_2^* , then $\mathbf{C}[C_1]$ is a localization of $\mathbf{C}[C_2]$ by all x^m for which $m \in C_2, m \cdot C_1^* = 0$. This allows us to construct inclusion maps between affine varieties \mathbf{A}_{C_i} and then glue them all together to form a toric variety $\mathbf{P} = \mathbf{P}_\Sigma$.

We already know how to describe sections of chiral de Rham complex on a flat space that corresponds to cone K^* of maximum dimension. Our next step is to construct vertex algebra that corresponds to a face of such cone. Namely, let C_1^* be generated by n_1, \dots, n_r where $n_1, \dots, n_{\dim N}$ form a basis of N and generate C^* . Then we can consider BRST operator

$$\mathcal{BRST}_g = \oint \sum_{i=1}^r g_i(n_i \cdot \Psi(z)) e^{\int n_i \cdot A_i(z)} dz$$

that acts on $\text{Fock}_{M \oplus C_1^*}$. Corresponding BRST cohomology will be denoted by $\mathcal{VA}_{C_1, g}$ and sections of chiral de Rham complex on \mathbf{A}_C will be denoted by $\mathcal{VA}_{C, g}$. We have a natural surjective map

$$\rho : \text{Fock}_{M \oplus C^*} \rightarrow \text{Fock}_{M \oplus C_1^*}$$

which commutes with \mathcal{BRST}_g . Here we, of course, abuse the notation a little bit by using two different definitions of \mathcal{BRST}_g for C^* and C_1^* . However, we assume that g_i there are the same for $i = 1, \dots, r$.

For every $m \in C$ and every $l \in \mathbf{Z}$ element $e^{\int m \cdot B}[l]$ acts on both $\mathcal{VA}_{C, g}$ and $\mathcal{VA}_{C_1, g}$. Really, its action on $\text{Fock}_{M \oplus C^*}$ commutes with \mathcal{BRST}_g , because $m \cdot C^* \geq$

0. Consider multiplicative system S generated by elements $e^{\int m \cdot B(z)}[0]$ with $m \in C$, $m \cdot C_1^* = 0$.

Proposition 7.1 Map ρ induces map

$$\rho_{BRST} : \mathcal{VA}_{C,g} \rightarrow \mathcal{VA}_{C_1,g}$$

which is precisely the localization map of $\mathbf{C}[C]$ -loop-module $\mathcal{VA}_{C,g}$ with respect to multiplicative system S .

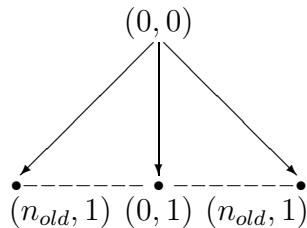
Proof. First we show that this map is the localization map of corresponding vector spaces with the action of the multiplicative system. For this it is enough to show that ρ is the localization map. This amounts to showing that any element v of $\text{Fock}_{M \oplus C}$ with eigen-values of $B[0]$ equal to n where $n \notin C_1^*$ is annihilated by some element in S . Since $n \notin C_1$, there exists an element $m \in C$ such that $m \cdot C_1 = 0$ and $m \cdot n > 0$. It is easy to see that a power of $s[0] = e^{\int m \cdot B}[0]$ annihilates v . Really, it does not change the $L[0]$ eigen-value of v but on the other hand it increases its $A[0] \cdot B[0]$ eigen-value by an arbitrary positive multiple of $m \cdot n$. So for big l the $L[0]$ eigen-value of $s[0]^l v$ is too small to fit into the subspace based on $|m_{\text{original}} + lm, n\rangle$, see Remark 6.1.

Since $\mathcal{VA}_{C,g}$ has structure of loop-module over $\mathbf{C}[C]$, its localization has structure of loop-module over $\mathbf{C}[C_1]$. One can also show that vertex algebra structure on $\mathcal{VA}_{C_1,g}$ is the localization of the structure on $\mathcal{VA}_{C,g}$. \square

Remark 7.2 It is interesting to observe that a surjective map on Fock spaces leads to an injective map on BRST cohomology.

Remark 7.3 Even though $e^{\int m \cdot B(z)}$ is invertible, its zero mode is not. This seems to contradict the calculations of Section 2 but the reason is the presence of negative $L[0]$ eigen-values in $\text{Fock}_{M \oplus K^*}$.

So we now have in our disposal a way of calculating sections of chiral de Rham complex of a smooth toric variety on any toric affine subset of it. This allows us to calculate cohomology of chiral de Rham complex. We will be most concerned with a calculation of cohomology of chiral de Rham complex for canonical line bundle L over a complete toric variety \mathbf{P} . To get the fan of L from the fan of \mathbf{P} one adds extra dimension to N and then lifts the fan of \mathbf{P} to height one as illustrated by the following figure.



We adjust our notations to denote the whole new lattice by $N = N_1 \oplus \mathbf{Z}$ and the new fan by Σ . An element in M that defines the last coordinate in N is denoted by "deg". Notice that for every cone $C^* \in \Sigma$ it is the same as "deg" from Proposition 6.6. We also denote by K^* the union of all cones in Σ , which may or may not be convex.

As it was noticed in Remark 5.3, we can consider quasi-loco sheaf $\pi_* \mathcal{MSV}(L)$ on \mathbf{P} . By Proposition 2.9, its cohomology could be calculated as Čech cohomology that corresponds to the covering of L by open affine subsets \mathbf{A}_C where we only consider the cones C^* that contain $(0, 1)$. So we need to consider Čech complex

$$0 \rightarrow \bigoplus_{C_0^*} \mathcal{VA}_{C_0, g} \rightarrow \bigoplus_{(C_0^*, C_1^*)} \mathcal{VA}_{C_{01}, g} \rightarrow \dots \rightarrow \bigoplus_{(C_0^*, \dots, C_r^*)} \mathcal{VA}_{C_{0\dots r}, g} \rightarrow 0$$

where $C_{0\dots k}$ is the dual of the intersection $C_{0\dots k}^*$ of C_0^*, \dots, C_k^* . Here we have chosen non-zero numbers g_n for all generators n of one-dimensional cones in Σ . We know that each $\mathcal{VA}_{C, g}$ is BRST cohomology of the corresponding Fock space and our goal is to write cohomology of Čech complex as certain BRST cohomology.

Proposition 7.4 Consider the following double complex

$$\begin{array}{ccc} 0 & \dots & 0 \\ \downarrow & \dots & \downarrow \\ 0 \rightarrow \bigoplus_{C_0^*} (\text{Fock}_{M \oplus C_0^*})_{\deg \cdot B[0]=0} & \rightarrow \dots \rightarrow & \bigoplus_{(C_0^*, \dots, C_r^*)} (\text{Fock}_{M \oplus C_{0\dots r}^*})_{\deg \cdot B[0]=0} \rightarrow 0 \\ \downarrow & \dots & \downarrow \\ 0 \rightarrow \bigoplus_{C_0^*} (\text{Fock}_{M \oplus C_0^*})_{\deg \cdot B[0]=1} & \rightarrow \dots \rightarrow & \bigoplus_{(C_0^*, \dots, C_r^*)} (\text{Fock}_{M \oplus C_{0\dots r}^*})_{\deg \cdot B[0]=1} \rightarrow 0 \\ \downarrow & \dots & \downarrow \\ \dots & \dots & \dots \end{array}$$

where vertical arrows are \mathcal{BRST}_g operators and horizontal arrows are sums of surjective maps of Fock spaces as dictated by definition of Čech cohomology. We also multiply vertical differentials in odd-numbered columns by (-1) to assure anticommutation of small squares. Then p -th cohomology of total complex is equal to $H^p(\pi_* \mathcal{MSV}(L), \mathbf{P})$. Here again we only consider cones C^* that contain $(0, \mathbf{R}_{\geq 0})$.

Proof. Proposition 6.6 tells us that cohomology along vertical lines happens only at the top (zeroth) row, where it becomes the Čech complex for the sheaf $\pi_* \mathcal{MSV}(L)$. So spectral sequence of one stupid filtration degenerates and converges to cohomology of $\pi_* \mathcal{MSV}(L)$. \square

Our next step is to calculate cohomology of total complex using the other stupid filtration. Let us see what happens if we take cohomology of horizontal maps of our double complex first. It could be done separately for each lattice element $n \in N$. If $\deg \cdot n = l$ then we are dealing with the l -th row. The part of the complex that we care about is a constant space $\text{Fock}_{M \oplus n}$ multiplied by a certain finite complex of vector spaces. That complex calculates cohomology of the simplex based on all indices i such that cones C_i^* that contain $(0, \mathbf{R}_{\geq 0})$ and n . If $n \notin K^*$ then the set is empty, and cohomology is zero. However, if $n \in K^*$ the cohomology is \mathbf{C} and is located at the zeroth column. As a result, horizontal cohomology is zero

except the zeroth column. Therefore, cohomology could be calculated by means of the restriction of \mathcal{BRST}_g operator applied to kernels of horizontal maps from the zeroth column. The following theorem describes this space.

Theorem 7.5 Consider the following degeneration of the vertex algebra structure on $\text{Fock}_{M \oplus K^*}$. In the definition of vertex operator $e^{\int (m \cdot B + n \cdot A)(z)}$ when applied to $|m_1, n_1\rangle$, the result is put to be zero, unless there is a cone in Σ that contains both n_1 and n . This does provide a consistent set of operator product expansions and the new algebra is denoted by $\text{Fock}_{M \oplus K}^\Sigma$. We denote by Δ^* the set of all generators of one-dimensional cones of Σ . We construct a BRST operator on $\text{Fock}_{M \oplus K}^\Sigma$ by the formula

$$\mathcal{BRST}_g = \oint \mathcal{BRST}_g(z) dz = \oint \sum_{n \in \Delta^*} g_n(n \cdot \Psi)(z) e^{\int n \cdot A(z)}.$$

Then we claim that $\oplus_p H^p(\pi_* \mathcal{MSV}(L))$ equals BRST cohomology of $\text{Fock}_{M \oplus K^*}^\Sigma$ with respect to \mathcal{BRST}_g .

Proof. In view of Proposition 7.4, it is enough to show that horizontal cohomology of the double complex of 7.4 at zeroth column and the corresponding vertical differential coincide with $\text{Fock}_{M \oplus K^*}^\Sigma$ and \mathcal{BRST}_g . Kernel of horizontal map consists of collections of elements of $\text{Fock}_{M \oplus C^*}$ that agree with restrictions. This can certainly be identified with $\text{Fock}_{M \oplus K^*}^\Sigma$ as follows. For every point $n \in N$ we take the corresponding n -part of the above collection of elements, since it is the same no matter which $C^* \ni n$ we choose. In the opposite direction, for each cone C^* we take a sum of n -parts for all n that belong to C^* . When we apply vertical arrows to such collections of elements, for each C^* we use only the part of \mathcal{BRST}_g that contains $B[0]$ eigen-values from that C^* . Under our identification this is precisely the action of the whole \mathcal{BRST}_g on $\text{Fock}_{M \oplus K^*}^\Sigma$ because as a result of that action for any $n \in C^*$ the only terms that survive and have a non-trivial projection back to C^* come from applying the part of \mathcal{BRST}_g with n in C^* . \square

We also want to show that the structure of vertex algebra induced on BRST cohomology of $\text{Fock}_{M \oplus K^*}^\Sigma$ coincides with the vertex algebra structure on cohomology of $\pi_* \mathcal{MSV}(L)$ defined in Proposition 3.7.

Proposition 7.6 Two structures of vertex algebra on $H^*(\pi_* \mathcal{MSV}(L))$ coincide.

Proof. The cup-product (n) is induced on Čech cohomology by the following product on Čech cochains. To define Čech differential we have chosen an order on the set of all cones. If $\alpha \in \mathcal{VA}_{C_0 \dots k, g}$ where $C_0 < C_1 < \dots < C_k$, and $\beta \in \mathcal{VA}_{C'_0 \dots l, g}$ where $C'_0 < C'_1 < \dots < C'_l$, then their (n) -product $\alpha_{(n)}\beta$ is zero unless

$$C_0 < \dots < C_k = C'_0 < \dots < C'_l,$$

in which case it is defined as the (n) -product of the restrictions of α and β to $\mathcal{VA}_{C_0 \dots k C'_1 \dots l, g}$. We extend this construction to define $\alpha_{(n)}\beta$ for any pair of elements of the double complex of Theorem 7.5 by replacing the (n) -product in $\mathcal{VA}_{C_0 \dots k C'_1 \dots l, g}$

by (n) -product in $\text{Fock}_{M \oplus (C_{0 \dots k}^* \cap C_{1 \dots l}')}.$ We now observe that for the differential d of the total complex we have

$$d(\alpha_{(n)}\beta) = (d\alpha)_{(n)}\beta + (-1)^{\text{parity}(\alpha) + \text{column}(\alpha)}\alpha_{(n)}(d\beta).$$

To check this we again use equation 4.6.9 of [13]. The product (n) induces the product on cohomology of vertical maps that is precisely the (n) product on Čech cochains of $\pi_*\mathcal{MSV}(L).$ It also induces a cup product on the cohomology of horizontal maps. A map between the two repeated cohomologies could be seen on the level of cochains as an addition of a coboundary, which is therefore compatible with $(n).$ It remains to notice that the (n) product on the zeroth column of our double complex simply acts as an independent application of (n) products for every cone $C^* \in \Sigma.$ So it coincides on the cohomology of the horizontal maps with the (n) product of the vertex algebra structure of $\text{Fock}_{M \oplus K^*}^\Sigma.$ \square

Remark 7.7 It is important to keep in mind that operations (n) do not define the structure of vertex algebra on the whole double complex, they only induce this structure on cohomology. This is analogous to the fact that the usual cup-product is not super-commutative on the level of cochains. Also, we can not really define a quasi-loco sheaf of vertex algebras over \mathbf{P} whose sections over \mathbf{A}_C are $\text{Fock}_{M \oplus C^*},$ because eigen-values of $L[0]$ are not bounded from below. Perhaps, it is just a matter of definitions, but localization might indeed behave poorly in this case.

Remark 7.8 It is clear that fields $L(z), J(z), G(z)$ and $Q(z)$ are still given by

$$\begin{aligned} Q(z) &= A(z) \cdot \Phi(z) - \deg \cdot \partial_z \Phi(z), \quad G(z) = B(z) \cdot \Psi(z), \\ J(z) &= : \Phi(z) \cdot \Psi(z) : + \deg \cdot B(z) \\ L(z) &= : B(z) \cdot A(z) : + : \partial_z \Phi(z) \cdot \Psi(z) : . \end{aligned}$$

Remark 7.9 Similar results can be obtained for cohomology of chiral de Rham complex for an arbitrary smooth toric variety, for example for a projective space. However in this paper we are mostly concerned with line bundle case because of its applications to Mirror Symmetry.

So now we have a good description of cohomology of chiral de Rham complex on a canonical bundle of a smooth toric variety. Our next step is to use the results of Section 5 to obtain a similar result for a Calabi-Yau hypersurface inside a smooth toric nef-Fano variety.

Certain combinatorial conditions on Σ are necessary to ensure that we have a Calabi-Yau hypersurface. Details could be found in the original paper of Batyrev [1]. The set Δ^* of all n should be the set of all lattice points of a convex polytope which we also denote by Δ^* abusing notations slightly. We do not require that all n

except $(0, 1)$ are vertices of Δ^* which geometrically means that the opposite of the canonical divisor on \mathbf{P} is nef but not necessarily ample. We also have a polytope $\Delta \in M$ defined as follows. Decomposition $N = N_1 \oplus \mathbf{Z}$ implies $M = M_1 \oplus \mathbf{Z}$. We define

$$\Delta = K \cap \{M_1, 1\}.$$

All vertices of polytope Δ belong to M . This is not entirely obvious, but follows from the fact that all cones of Σ are basic and therefore Δ is reflexive in M_1 . A Calabi-Yau hypersurface in \mathbf{P} is given by a section of the negative canonical line bundle of \mathbf{P} . Any such section is given by a set of numbers f_m , one for each lattice point m in Δ . If f is generic, the resulting hypersurface in \mathbf{P} is smooth and Calabi-Yau. In what follows we will denote $(0, 1)$ by \deg^* .

Let us fix a generic section μ of the anti-canonical line bundle of \mathbf{P} and hence a function $f : \Delta \rightarrow \mathbf{C}$.

Proposition 7.10 Operator $\mathcal{BRST}_\mu(z)$ from Lemma 5.5 is given by

$$\mathcal{BRST}_\mu(z) = \sum_{m \in \Delta} f_m(m \cdot \Phi)(z) e^{\int m \cdot B(z)}.$$

Proof. It is enough to consider a section $\mu = f_m x^m$. For any maximum cone of Σ with basis $(n_1, \dots, n_{\dim N})$ we see that

$$\begin{aligned} \mathcal{BRST}_\mu(z) &= f_m \sum_i \varphi_i(m \cdot n_i) e^{\int (m - m_i) \cdot B(z)} = f_m \sum_i (m_i \cdot \Phi)(z) (m \cdot n_i) e^{\int m \cdot B(z)} \\ &= f_m(m \cdot \Phi)(z) e^{\int m \cdot B(z)}. \end{aligned}$$

□

From now on we denote \mathcal{BRST}_μ by \mathcal{BRST}_f . We also denote by X the Calabi-Yau hypersurface in \mathbf{P} which is given by f . We will denote by X_C the intersection of X with \mathbf{A}_C .

Proposition 7.11 For every cone $C^* \in \Sigma$ sections of $\mathcal{MSV}(X)$ on X_C are given by BRST cohomology of Fock $_{M \oplus C^*}$ by the operator

$$\mathcal{BRST}_{f,g} = \oint \mathcal{BRST}_{f,g}(z) dz$$

where

$$\begin{aligned} \mathcal{BRST}_{f,g}(z) &= \mathcal{BRST}_f(z) + \mathcal{BRST}_g(z) \\ &= \sum_{m \in \Delta} f_m(m \cdot \Phi)(z) e^{\int m \cdot B(z)} + \sum_{n \in \Delta^* \cap C^*} g_n(n \cdot \Psi)(z) e^{\int n \cdot A(z)}. \end{aligned}$$

Proof. One easily computes that all modes of $\mathcal{BRST}_f(z)$ and $\mathcal{BRST}_g(z)$ anti-commute with each other. Also, Proposition 5.7 implies that sections of $\mathcal{MSV}(X)$

are cohomology with respect to \mathcal{BRST}_f of cohomology of $\text{Fock}_{M \oplus C^*}$ with respect to \mathcal{BRST}_g . Consider the following double complex.

$$\begin{array}{ccccccc}
& \dots & 0 & 0 & 0 & \dots & \\
& & \downarrow & \downarrow & \downarrow & & \\
& \dots \rightarrow & \text{Fock}_{-1,0} & \rightarrow & \text{Fock}_{0,0} & \rightarrow & \text{Fock}_{1,0} \rightarrow \dots \\
& & \downarrow & \downarrow & \downarrow & & \\
& \dots \rightarrow & \text{Fock}_{-1,1} & \rightarrow & \text{Fock}_{0,1} & \rightarrow & \text{Fock}_{1,1} \rightarrow \dots \\
& & \downarrow & \downarrow & \downarrow & & \\
& \dots \rightarrow & \text{Fock}_{-1,2} & \rightarrow & \text{Fock}_{0,2} & \rightarrow & \text{Fock}_{1,2} \rightarrow \dots \\
& & \downarrow & \downarrow & \downarrow & & \\
& \dots & \dots & \dots & \dots & &
\end{array}$$

where $\text{Fock}_{k,l}$ is a shorthand for the part of $\text{Fock}_{M \oplus C^*}$ where $(\deg^* \cdot A)[0]$ and $(\deg \cdot B)[0]$ equal k and l respectively. Horizontal maps are \mathcal{BRST}_f and vertical maps are \mathcal{BRST}_g . We already know that columns of this double complex are exact everywhere except the zeroth row. A standard diagram chase then implies that horizontal cohomology of zeroth kernels of vertical maps are isomorphic to cohomology of total complex. However, total complex and differential on it are precisely $\text{Fock}_{M \oplus C^*}$ and $\mathcal{BRST}_{f,g}$. \square

Proposition 7.12 In the above proposition all cohomology of total complex are trivial, except for the zeroth one.

Proof. Grading by $\deg^* \cdot A[0]$ on $\pi_* \mathcal{MSV}(L)$ corresponds to counting $\#(b^1) + \#(\varphi^1) - \#(a_1) - \#\psi_1$ where x^1 is the special coordinate of the line bundle. Since this count is zero for $\mathcal{MSV}(X)$, the cohomology of \mathcal{BRST}_f is concentrated at zeroth column. \square

Remark 7.13 Above identification is also compatible with vertex algebra structures. Really, this structure is induced from that of $\text{Fock}_{M \oplus C^*}$ both for the repeated and for the single use of BRST cohomology.

Now we are in position to calculate cohomology of chiral de Rham complex of Calabi-Yau hypersurfaces in toric Fano varieties.

Theorem 7.14 BRST cohomology of $\text{Fock}_{M \oplus K^*}^\Sigma$ with respect to BRST operator $\mathcal{BRST}_{f,g}$ equals $H^*(X, \mathcal{MSV}(X))$.

Proof. Argument is completely analogous to that of Proposition 7.4. We construct a double complex similar to that of 7.4, but with \mathcal{BRST}_g changed to $\mathcal{BRST}_{f,g}$ and $\deg \cdot B[0]$ changed to $\deg \cdot B[0] + \deg^* \cdot A[0]$. Proposition 7.12 assures that spectral sequence of this double complex degenerates. \square

It is now a technical matter to calculate fields $L(z)$, $J(z)$, $G(z)$ and $Q(z)$.

Proposition 7.15 Fields L , J , G , and Q on $H^*(\mathcal{MSV}(X))$ are induced from the following fields on $\text{Fock}_{M \oplus K^*}^\Sigma$

$$\begin{aligned} Q(z) &= A(z) \cdot \Phi(z) - \deg \cdot \partial_z \Phi(z) \\ G(z) &= B(z) \cdot \Psi(z) - \deg^* \cdot \partial_z \Psi(z) \\ J(z) &= : \Phi(z) \cdot \Psi(z) : + \deg \cdot B(z) - \deg^* \cdot A(z) \\ L(z) &= : B(z) \cdot A(z) : + : \partial_z \Phi(z) \cdot \Psi(z) : - \deg^* \cdot \partial_z A(z). \end{aligned}$$

Proof. By a standard application of Wick theorem, one observes that the above operators satisfy OPEs of the topological algebra of dimension $(\dim N - 2)$. Then it remains to show that $G(z)$ and $Q(z)$ are correct. This follows from Proposition 5.8 and Remark 7.8. \square

Remark 7.16 Notice that OPEs of topological algebra hold exactly in $\text{Fock}_{M \oplus K^*}^\Sigma$ even though we only need them to hold modulo the image of $\mathcal{BRST}_{f,g}$. This algebra was discovered almost three years ago as a lucky guess motivated by Mirror Symmetry. The above algebra behaves exactly like the holomorphic part of $N = (2, 2)$ theory under mirror involution, see for example [22]. This becomes apparent when we undo the topological twist and consider $N = 2$ super-conformal algebra with

$$\begin{aligned} L_{N=2}(z) &= : B(z) \cdot A(z) : + : (1/2)(\partial_z \Phi(z) \cdot \Psi(z) - \Phi \cdot \partial_z \Psi(z)) : \\ &\quad - (1/2)\deg^* \cdot \partial_z A(z) - (1/2)\deg \cdot \partial B(z). \end{aligned}$$

Mirror Symmetry in this setup means switching M and N , Δ and Δ^* , \deg and \deg^* , f and g . We want to show that A model vertex algebra of a Calabi-Yau hypersurface equals B model algebra of its mirror. Certainly the formulas above show that this is true for the operators G , Q , J and L . However, there are some obstacles that prevent us from making such a statement. The easiest objection is that the cocycle used to make vertex operators bosonic does not look symmetric. However, that could be fixed by noticing that multiplication of $|m, n\rangle$ by $(-1)^{m \cdot n}$ is equivalent to switching the roles of M and N in the definition of the cocycle. We also notice that when we switch M and N , and consequently Ψ and Φ , the grading of the operators changes slightly. Besides, the Fock space we consider is based on $M \oplus K^*$, rather than on $M \oplus N$ or $K \oplus K^*$. Both these objections will be addressed successfully in the next section. However, there is one more difficulty that can not be resolved: we use the subdivision of the cone K^* but we do not subdivide lattice M at all. This will eventually lead to the following statement

A and B model vertex algebras of mirror symmetric Calabi-Yau hypersurfaces are different degenerations of two families of topological vertex algebras that are related by a mirror involution.

8 Transition to the whole lattice

The goal of this section is to show that cone K^* in Theorem 7.14 could be replaced by the whole lattice N . This is certainly far from obvious. We construct and use certain homotopy operators whose anticommutators with $\mathcal{BRST}_{f,g}$ are identity plus operators that "push elements closer to K^* ". Before we start, notice that decomposition of K^* into cones can be extended to decomposition of lattice N by adding arbitrary multiples of \deg^* to all cones. This allows us to define vertex algebra $\text{Fock}_{M \oplus N}^\Sigma$ analogously to the definition of the vertex algebra $\text{Fock}_{M \oplus K^*}^\Sigma$.

Proposition 8.1 For every vertex m_0 of Δ there is an operator \mathcal{R}_{m_0} such that

$$\mathcal{R}_{m_0} \mathcal{BRST}_{f,g} + \mathcal{BRST}_{f,g} \mathcal{R}_{m_0} = \mathbf{1} + \alpha$$

where α strictly increases eigen-values of $m_0 \cdot B[0]$ and does not decrease eigen-values of $m \cdot B[0]$ for any other $m \in \Delta$.

Proof. Consider graded ring $\mathbf{C}[K]$. Pick a basis $n_1, \dots, n_{\dim N}$ of N . It was proved in [6] that for general values of f_m elements $\sum_m f_m x^m (m \cdot n_i)$ form a regular sequence. In particular, quotient ring is Artinian and for a sufficiently big k ($k = \dim M$ is in fact always enough) element x^{km_0} lies in the ideal generated by the above regular sequence. So we have

$$x^{km_0} = \sum_i h_i(x^\Delta) \sum_m f_m x^m (m \cdot n_i).$$

We now consider the following field

$$\mathcal{R}_{m_0}(z) = e^{-\int km_0 \cdot B(z)} \sum_i h_i(e^{\int \Delta \cdot B}) n_i \cdot \Psi(z).$$

We have the operator product expansion

$$\mathcal{R}_{m_0}(z) \mathcal{BRST}_f(w) \sim \frac{1}{z-w} e^{-\int km_0 \cdot B} \sum_i h_i(e^{\int \Delta \cdot B}) \sum_m f_m e^{\int m \cdot B} (m \cdot n_i) = \frac{1}{z-w}.$$

With the usual abuse of notation, we introduce

$$\mathcal{R}_{m_0} = \oint \mathcal{R}_{m_0}(z) dz.$$

We now argue that this operator satisfies the claim of this proposition. First of all, the above OPE shows us that

$$\mathcal{R}_{m_0} \mathcal{BRST}_f + \mathcal{BRST}_f \mathcal{R}_{m_0} = \mathbf{1}.$$

Let us look at its anticommutator with \mathcal{BRST}_g . Operator product expansion of $\mathcal{R}_{m_0}(z) e^{\int n \cdot A(w)} n \cdot \Psi(w)$ is non-singular if $m_0 \cdot n = 0$. Otherwise, fields in the OPE shift eigen-values of $m_0 \cdot B[0]$ positively (more precisely by $m_0 \cdot n$). Of course, for any other $m \in \Delta$ eigen-values of $m \cdot B[0]$ are shifted by $m \cdot n$ which is non-negative. \square

The above proposition provides us with necessary tools to prove the main result of this section.

Proposition 8.2 Cohomology of $\text{Fock}_{M \oplus K^*}^\Sigma$ with respect to $\mathcal{BRST}_{f,g}$ is isomorphic to cohomology of $\text{Fock}_{M \oplus N}^\Sigma$ with respect to $\mathcal{BRST}_{f,g}$.

Proof. It is clear that $\mathcal{BRST}_{f,g} \text{Fock}_{M \oplus K^*}^\Sigma \subseteq \text{Fock}_{M \oplus K^*}^\Sigma$. Then one needs to prove the following two inclusions

- $\text{Ker}(\text{Fock}_{M \oplus K^*}^\Sigma) \cap \text{Im}(\text{Fock}_{M \oplus N}^\Sigma) \subseteq \text{Im}(\text{Fock}_{M \oplus K^*}^\Sigma);$
- $\text{Ker}(\text{Fock}_{M \oplus K^*}^\Sigma) + \text{Im}(\text{Fock}_{M \oplus N}^\Sigma) \supseteq \text{Ker}(\text{Fock}_{M \oplus K^*}^\Sigma).$

Notice that the opposite inclusions are obvious.

First inclusion. Assume that there exists an element $v \in \text{Fock}_{M \oplus K^*}^\Sigma$ such that $v = \mathcal{BRST}_{f,g} v_1$ where v_1 does not lie in $\text{Fock}_{M \oplus K^*}^\Sigma$. Moreover, of all such v_1 we pick the one which is "the closest to $\text{Fock}_{M \oplus K^*}^\Sigma$ ". The distance is defined as follows. We look at all codimension one faces of K^* or equivalently all vertices of Δ . For every vertex m of Δ we look at the maximum eigen-value of $-m \cdot B[0]$ on components of v_1 . We call the maximum of this number and zero the m -distance from v_1 to $\text{Fock}_{M \oplus K^*}^\Sigma$. Then the total distance from v_1 to $\text{Fock}_{M \oplus K^*}^\Sigma$ is the sum of m -distances for all vertices m of Δ . So we pick v_1 with a minimum distance and our goal is to show that this distance is zero.

If the distance is not zero, then for one of the vertices m there is a component of v_1 with a negative eigen-value of $m \cdot B[0]$. We now apply the result of Proposition 8.1. Consider operator \mathcal{R}_m . We have

$$(\mathcal{R}_m \mathcal{BRST}_{f,g} + \mathcal{BRST}_{f,g} \mathcal{R}_m)v = v + \alpha v.$$

So

$$v = \mathcal{BRST}_{f,g}(\mathcal{R}_m v + \alpha v_1),$$

because α commutes with $\mathcal{BRST}_{f,g}$. Notice now that $\mathcal{R}_m v \in \text{Fock}_{M \oplus K^*}^\Sigma$ and the distance from αv_1 to $\text{Fock}_{M \oplus K^*}^\Sigma$ is strictly less than the distance from v_1 to it. Really, the m -distance is smaller, and m_1 -distance is not bigger for any other vertex m_1 of Δ . This contradicts minimality of v_1 .

Second inclusion. Our argument here is similar. Let v be an element of $\text{Fock}_{M \oplus N}^\Sigma$ such that $\mathcal{BRST}_{f,g} v = 0$. Then for every vertex $m \in \Delta$ we have

$$\mathcal{BRST}_{f,g} \mathcal{R}_m v = v + \alpha_m v$$

and therefore

$$v \equiv -\alpha_m v \pmod{\text{Im}(\text{Fock}_{M \oplus N}^\Sigma)}.$$

By applying α_m for different m sufficiently many times, we can again push v into $\text{Fock}_{M \oplus K^*}^\Sigma$. \square

We now combine Propositions 8.2 and 7.15 with Theorem 7.14 to formulate one of the main results of this paper.

Theorem 8.3 Let X be a Calabi-Yau hypersurface in a smooth toric nef-Fano variety, given by $f : \Delta \rightarrow \mathbf{C}$ and a fan Σ . Then cohomology of chiral de Rham complex of X equals BRST cohomology of $\text{Fock}_{M \oplus N}^\Sigma$ by operator

$$\mathcal{BRST}_{f,g} = \oint \left(\sum_{m \in \Delta} f_m(m \cdot \Phi)(z) e^{\int m \cdot B(z)} + \sum_{n \in \Delta^*} g_n(n \cdot \Psi)(z) e^{\int n \cdot A(z)} \right) dz$$

with any choice of non-zero numbers g_n . Additional structure of topological vertex algebra is given by formulas of Proposition 7.15.

We have therefore addressed one of the questions posed at the end of last section. Another obstacle for Mirror Symmetry stated there was the fact that the modes of Φ and Ψ are defined differently. However, this is precisely what happens when we go from A model to B model. Because $J =: \Phi \cdot \Psi : + \deg \cdot B - \deg^* \cdot A$, the moding of Ψ and Φ changes when we go from $L_{A\text{-model}}[0]$ to $L_{B\text{-model}}[0] = L_{A\text{-model}}[0] + J[0]$. Really, while $\Phi[k]$ and $\Psi[k]$ change, the true modes $\Phi_{(k)}$ and $\Psi_{(k)}$ are not affected by the switch of the roles M and N .

It remains to address the following question. What is the real meaning of going from $\text{Fock}_{M \oplus N}$ to $\text{Fock}_{M \oplus N}^\Sigma$? It turns out that in the case when Σ admits a convex piece-wise linear function (which geometrically means that \mathbf{P} is projective) this vertex algebra is a degeneration of vertex algebra $\text{Fock}_{M \oplus N}$.

The degeneration we are about to describe is completely analogous to the one discussed in [6] but is now performed for the whole Fock space. Let $h : N_{\mathbf{R}} \rightarrow \mathbf{R}$ be a continuous function which is linear on every cone of Σ and satisfies

$$h(x + y) \leq h(x) + h(y)$$

with equality achieved if and only if x and y lie in the same cone of Σ . Then we get ourselves a complex parameter t and start changing the basis of $\text{Fock}_{M \oplus N}$ by assigning $|m, n\rangle_t = t^{h(n)} |m, n\rangle$. To preserve the definition of vertex algebra we also multiply $e^{\int n \cdot A(z)}$ by $t^{h(n)}$. Now if we let t go to zero, the structure of vertex algebra of $\text{Fock}_{M \oplus N}$ will go to the structure of vertex algebra of $\text{Fock}_{M \oplus N}^\Sigma$. When the structure of $\text{Fock}_{M \oplus N}^\Sigma$ is defined via this limit, we can also get the action of $\mathcal{BRST}_{f,g}$ on it as a limit of

$$\mathcal{BRST}_{f,g}(t) = \oint \left(\sum_{m \in \Delta} f_m(m \cdot \Phi)(z) e^{\int m \cdot B(z)} + \sum_{n \in \Delta^*} g_n t^{h(n)} (n \cdot \Psi)(z) e^{\int n \cdot A(z)} \right) dz$$

This prompts the following definition.

Definition 8.4 We define *Master Family of vertex algebras* that corresponds to the pair of reflexive polytopes Δ and Δ^* as the BRST quotient of the vertex algebra $\text{Fock}_{M \oplus N}$ by the operator

$$\mathcal{BRST}_{f,g} = \oint \left(\sum_{m \in \Delta} f_m(m \cdot \Phi)(z) e^{\int m \cdot B(z)} + \sum_{n \in \Delta^*} g_n (n \cdot \Psi)(z) e^{\int n \cdot A(z)} \right) dz$$

where f and g are parameters of the theory. Additional structure of topological vertex algebra is given by formulas of Proposition 7.15.

Conjecture 8.5 Vertex algebras that appear in Mirror Symmetry for hypersurfaces defined by Δ and Δ^* are degenerations of Master Family of vertex algebras.

Remark 8.6 Large complex structure limit (see [18]) in our language is most likely the double degeneration of Master Family where both M and N are subdivided. This theory is a degeneration of theories on both mirror manifolds and can also be used to link the two.

Our discussion so far have been focused around reflexive polytopes Δ^* that admit a unimodular triangulation and therefore yield smooth \mathbf{P} s. This is a very important class of examples, which includes famous quintic in \mathbf{P}^4 , but most reflexive polytopes do not fall into this category. Next two sections will be devoted to the treatment of singular \mathbf{P} s. We can no longer use the definition of [17], but many of our results still hold in that generality under appropriate definitions.

9 Vertex algebras of Gorenstein toric varieties

The goal of this section is to define an analog of chiral de Rham complex for an arbitrary Gorenstein toric variety. It is again a loco sheaf of conformal vertex algebras. Sections of this sheaf over any toric affine chart admit a structure of topological vertex algebra which may or may not be compatible with the localization. However, $J[0]$ and $Q[0]$ are globally defined, which allows us to introduce string de Rham complex and to propose a definition of string cohomology vector spaces. Recall that dimensions of these spaces were rigorously defined by Batyrev and Dais in [5] but the spaces themselves have never been constructed mathematically.

We are working in the following setup. There are dual lattices M and N with a primitive element "deg" fixed in M . There is a fan Σ in N such that all generators n_i of its one-dimensional faces satisfy $\deg \cdot n_i = 1$. A set Δ^* consists of some lattice points of degree one inside the union of all cones of Σ . We do not generally require that Δ^* includes all such points, or that it is a set of all lattice points inside a convex polytope. However, we do demand that it contains generators of all one-dimensional cones of Σ . At last, we have a generic set of numbers g_n for all $n \in \Delta^*$.

Definition 9.1 For each cone $C^* \in \Sigma$ we denote by $\mathcal{V}_g(C)$ the BRST cohomology of the vertex algebra $\text{Fock}_{M \oplus C^*}$ with respect to the BRST operator

$$\mathcal{BRST}_g = \oint \left(\sum_{n \in \Delta^* \cap C^*} g_n (n \cdot \Psi)(z) e^{\int n \cdot A(z)} \right) dz.$$

We also provide this algebra with structure of topological algebra by introducing operators

$$Q(z) = A(z) \cdot \Phi(z) - \deg \cdot \partial_z \Phi(z), \quad G(z) = B(z) \cdot \Psi(z),$$

$$J(z) = : \Phi(z) \cdot \Psi(z) : + \deg \cdot B(z),$$

$$L(z) = : B(z) \cdot A(z) : + : \partial_z \Phi(z) \cdot \Psi(z) : .$$

Remark 9.2 Above definition does not guarantee that the resulting vertex algebra has no negative eigen-values of $L[0]$. To prove this and much more we first consider the case of simplicial cone C^* . Then we extend our results to the general case by looking at the degeneration of $\text{Fock}_{M \oplus C^*}$ that corresponds to subdivision of C^* into simplicial cones.

First of all, we consider the case of orbifold singularities in which case we can give an explicit description of BRST cohomology similar to that of Proposition 6.6. When we talk about orbifold singularities we implicitly assume that not only the cones C^* are simplicial, but also g_n are zero, except for the generators of one-dimensional faces of C^* .

Proposition 9.3 Let C^* be a simplicial cone of dimension $\dim N$. Its faces of dimension one are generated by $n_1, n_2, \dots, n_{\dim C^*}$. Denote by N_{small} the sublattice of N generated by n_i . Denote by M_{big} the suplattice of M which is the dual of N_{small} . Let the dual of C^* in M_{big} be generated by $m_1, \dots, m_{\dim M}$. For every i we define

$$b^i(z) = e^{\int m_i \cdot B(z)}, \quad \varphi^i(z) = (m_i \cdot \Phi(z)) e^{\int m_i \cdot B(z)}, \quad \psi_i(z) = (n_i \cdot \Psi(z)) e^{-\int m_i \cdot B(z)},$$

$$a_i(z) = : (n_i \cdot A(z)) e^{-\int m_i \cdot B(z)} : + : (m_i \cdot \Phi(z)) (n_i \cdot \Psi(z)) e^{-\int m_i \cdot B(z)} :$$

for all $i = 1, \dots, \dim M$. These fields generate a vertex subalgebra $\mathcal{VA}_{C^*, M_{big}}$ inside $\text{Fock}_{M_{big} \oplus 0}$. Consider all fields from $\mathcal{VA}_{C^*, M_{big}}$ whose $A[0]$ eigen-values lie in M . Denote the resulting algebra by $\mathcal{VA}_{C^*, M}$. Let $\text{Box}(C^*)$ be the set of all elements in $n \in C^*$ such that $(n - n_i) \notin C^*$ for all i . For every $n \in \text{Box}(C^*)$ consider the following set of elements of $\text{Fock}_{M \oplus n}$. For every $v_0 = \prod A[\dots] \prod B[\dots] \prod \Phi[\dots] \prod \Psi[\dots] |m, 0\rangle$ that lies in $\mathcal{VA}_{C^*, M}$ consider $v = \prod A[\dots] \prod B[\dots] \prod \Phi[\dots] \prod \Psi[\dots] |m, n\rangle$ which is obtained by applying the same modes of A, B, Φ and Ψ to $|m, n\rangle$ instead of $|m, 0\rangle$. We denote this space by $\mathcal{VA}_{C^*, M}^{(n)}$. Then we claim that

$$\mathcal{V}_g(C) = \bigoplus_{n \in \text{Box}(C^*)} \mathcal{VA}_{C^*, M}^{(n)}.$$

Proof. First of all, the argument of Proposition 6.5 shows that

$$\mathcal{V}_g(C) = \bigoplus_{n \in \text{Box}(C^*)} \text{Ker}(\mathcal{BRST}_g : \text{Fock}_{M \oplus n} \rightarrow \text{Fock}_{M \oplus C^*}).$$

Then we notice that changing $|m, n\rangle$ to $|m, 0\rangle$ commutes with the action of \mathcal{BRST}_g so it is enough to concern ourselves with the case $n = 0$. Then the kernel of \mathcal{BRST}_g on $\text{Fock}_{M \oplus 0}$ is the intersection of $\text{Fock}_{M \oplus 0}$ with the kernel of \mathcal{BRST}_g on $\text{Fock}_{M_{big} \oplus 0}$. It remains to apply Proposition 6.6. \square

Remark 9.4 Corresponding space \mathbf{A}_C is a quotient of a flat space by an abelian group. The part at $n = 0$ is precisely the invariant part of the flat space algebra, while other n correspond to "twisted sectors".

Proposition 9.5 For a simplicial cone C^* of dimension $\dim N$ eigen-values of $L[0]$ on $\mathcal{V}_g(C)$ are non-negative. Eigen-values of $A[0]$ on the zero eigen-space of $L[0]$ lie in C . Besides, for any $d > 0$, eigenvalues of $A[0]$ on $L[0] = d$ eigen-space lie in $C - D(d) \deg$ where $D(d)$ is some constant which depends only on d and dimension of N .

Proof. Consider $n \in \text{Box}(C^*)$ given by

$$n = \sum_i \alpha_i n_i.$$

Let us consider all elements v_0 from $\mathcal{VA}_{C^*, M_{big}}$ and the corresponding elements v obtained by changing $|m, 0\rangle$ to $|m, n\rangle$. Such a change incurs a change in $L[0]$ which is equal to $m \cdot n$, and is therefore linear in m . Hence, we can assess new $L[0]$ by adding α_i for each occurrence of b^i or φ^i and subtracting α_i for each occurrence of a_i or ψ_i . So to show that $L[0]$ has no negative eigen-values, one has to show that each mode of a_i , b^i , φ^i or ψ_i contributes non-negatively. According to Proposition 6.6, we need to consider non-positive modes of b and φ and negative modes of a and ψ . The resulting contributions are collected in the following table.

mode	$a_i[-k]$	$b^i[-k]$	$\varphi^i[-k]$	$\psi_i[-k]$
$\Delta L[0]$	$k - \alpha_i$	$k + \alpha_i$	$k + \alpha_i$	$k - \alpha_i$

Since $0 \leq \alpha_i < 1$, all entries are non-negative, which proves the first part of the proposition. Moreover, the only time a zero entry can occur is when $\alpha_i = 0$ and we are looking at $b^i[0]$ or $\varphi^i[0]$, which proves the second part. Finally, for a fixed d there are only finitely many ways to combine $a[-k]$, $b[-k]$, $\varphi[-k]$ and $\psi[-k]$ to get $L[0] = d$, up to arbitrary extra $b^i[0]$ and $\varphi^i[0]$. This finishes the proof. \square

We now extend these results to simplicial cones of dimension smaller than $\dim N$.

Proposition 9.6 For any simplicial cone C^* eigen-values of $L[0]$ on $\mathcal{V}_g(C)$ are non-negative. Eigen-values of $A[0]$ on the zero eigen-space of $L[0]$ lie in C . Besides, for any $d > 0$, eigenvalues of $A[0]$ on $L[0] = d$ eigen-space lie in $C - D(d) \deg$ where $D(d)$ is some constant which depends only on d and dimension of N .

Proof. Consider an arbitrary simplicial cone C_1^* of maximum dimension whose one-dimensional faces are generated by elements in Δ^* such that C^* is a face of C_1^* . Proposition 9.5 implies that \mathcal{V}_{C_1} is $\mathbf{C}[C_1]$ -loop-module. Proof of Proposition 7.1 is applicable in this more general situation and allows us to show that \mathcal{V}_C is a localization of \mathcal{V}_{C_1} with respect to the multiplicative system $S = \{x^m, m \cdot C^* = 0, m \cdot C_1^* \geq 0\}$. Grading operator $L[0]$ is still non-negative on the localization, and $L[0] = 0$ part of \mathcal{V}_C is the localization of $L = 0$ part of \mathcal{V}_{C_1} . It remains to observe that for every $x^m \in S$ we have $-m \in C$ so localization does not push $A[0]$ eigen-values from C . \square

We are now in position to drop simpliciality assumption on cone C^* . This will require a careful investigation of degeneration of vertex algebras given by a triangulation of a non-simplicial cone.

Proposition 9.7 For any cone C^* and a generic choice of g , all eigen-values of $L[0]$ on \mathcal{V}_C are non-negative.

Proof. Consider an arbitrary regular triangulation of $C^* \cap \Delta^*$ and corresponding decomposition of C^* into union of simplicial cones. Denote by $\text{Fock}_{M \oplus C^*}^{\text{triang}}$ degeneration of vertex algebra $\text{Fock}_{M \oplus C^*}$ as in Theorem 7.5. As in Proposition 7.4 we consider the double complex

$$\begin{array}{ccccccc}
& 0 & & \dots & & 0 & \\
& \downarrow & & \dots & & \downarrow & \\
0 \rightarrow \bigoplus_{C_0^*} (\text{Fock}_{M \oplus C_0^*})_{\deg \cdot B[0]=0} & \rightarrow \dots \rightarrow & \bigoplus_{(C_0^*, \dots, C_r^*)} (\text{Fock}_{M \oplus C_{0 \dots r}^*})_{\deg \cdot B[0]=0} & \rightarrow 0 \\
& \downarrow & & \dots & & \downarrow & \\
0 \rightarrow \bigoplus_{C_0^*} (\text{Fock}_{M \oplus C_0^*})_{\deg \cdot B[0]=1} & \rightarrow \dots \rightarrow & \bigoplus_{(C_0^*, \dots, C_r^*)} (\text{Fock}_{M \oplus C_{0 \dots r}^*})_{\deg \cdot B[0]=1} & \rightarrow 0 \\
& \downarrow & & \dots & & \downarrow & \\
& \dots & & \dots & & \dots & \\
\end{array}$$

where C_0^* are all cones in the triangulation. Again we consider two spectral sequences associated to this double complex. When you take horizontal maps first, the only non-trivial cohomology appears at zeroth column. Moreover, when you apply vertical cohomology to zeroth kernels of horizontal maps, you get BRST cohomology of $\text{Fock}_{M \oplus C^*}^{\text{triang}}$ by \mathcal{BRST}_g . This shows that cohomology of the total complex is isomorphic to BRST cohomology of $\text{Fock}_{M \oplus C^*}^{\text{triang}}$. The other stupid filtration implies that there exists a spectral sequence that converges to BRST cohomology of $\text{Fock}_{M \oplus C^*}^{\text{triang}}$ and starts with $\bigoplus_{C_{0 \dots k}} \mathcal{V}_g(C_{0 \dots k})$. Everything in this picture is additionally graded by eigen-values of $L[0]$. Proposition 9.6 shows that $\mathcal{V}_g(C_{0 \dots k})$ have no negative eigen-values of $L[0]$, therefore BRST cohomology of $\text{Fock}_{M \oplus C^*}^{\text{triang}}$ can not have any negative eigen-values either.

Now it remains to go from BRST cohomology of $\text{Fock}_{M \oplus C^*}^{\text{triang}}$ to that of $\text{Fock}_{M \oplus C^*}$. Notice that $A[0]$, $L[0]$ and $J[0]$ commute with each other and with \mathcal{BRST}_g so we can consider separately the parts of $\text{Fock}_{M \oplus C^*}$ and $\text{Fock}_{M \oplus C^*}^{\text{triang}}$ that have fixed eigenvalues of $A[0]$, $J[0]$ and $L[0]$. We can show that these spaces are finite-dimensional. Really, for a fixed eigenvalue m of $A[0]$ we can find an integer r such that $m + r \deg$ lies in C . Then we claim that all eigen-spaces of $L[0] + (r+1)J[0]$ on $\text{Fock}_{m \oplus C^*}$ are finite-dimensional. For each n we start with at least $\deg \cdot n$ and almost all modes of A , B , Φ and Ψ can only increase this eigenvalue. The exception is a few modes of Φ or Ψ , but they are fermionic and can only appear in a finite number of combinations. This proves that all eigen-spaces of $L[0] + (r+1)J[0]$ are finite-dimensional, and so are all spaces with fixed $A[0]$, $L[0]$ and $J[0]$ eigen-values.

Now \mathcal{BRST}_g for $\text{Fock}_{M \oplus C^*}^{\text{triang}}$ can be seen as the true limit of operators $\mathcal{BRST}_{g(t)}$ as discussed right before the Definition 8.4. For families of operators on finite dimensional spaces, dimensions of kernels jump at special points and dimensions of images decrease, so dimensions of cohomology jump. Since for all negative $L[0]$ there is no cohomology for the degenerate map, there is no cohomology for the original $\text{Fock}_{M \oplus C^*}$ for generic (that is outside of countably many Zariski closed subsets of codimension one) choices of g . \square

We can use similar arguments to extend the rest of the results of Propositions 9.5 and 9.6 to arbitrary Gorenstein cones C^* .

Proposition 9.8 For any cone C^* , the eigen-values of $A[0]$ on the zero eigen-space of $L[0]$ lie in C . Besides, for any $d > 0$, eigenvalues of $A[0]$ on $L[0] = d$ eigen-space lie in $C - D(d) \deg$ where $D(d)$ is some constant which depends only on d and dimension of N .

Proof. For $L[0] = 0$, it is enough to show for every generator n of a one-dimensional face of C^* that eigen-values of $n \cdot A[0]$ are non-negative on the zero eigen-space of $L[0]$. Argument of the above proposition shows that it is enough to produce a regular triangulation of C^* such that all its maximum cones contain n . Really, we can then apply the second part of Proposition 9.6 and the same degeneration argument works. Similarly, to show that for $L[0] = d$ all eigenvalues of $A[0]$ lie in $C - D(d) \deg$ means to show that $n \cdot A[0] \geq -D(d)$ for every generator n of a one-dimensional face of C^* . Again if we can produce a regular triangulation such that all of its maximum cones contain n , then we can use the same degeneration argument and Proposition 9.6.

To construct such a triangulation, we do the following. We consider polytope $P = C^* \cap \Delta^*$. For every vertex n_i of P we move it slightly away from n along the line from n to n_i . For small generic perturbation of this type, the resulting (non-integer) points n_i and n will still be a set of vertices of a convex polytope P' . All faces of P' that do not contain n will be simplicial, and we will call the union of these faces the outer surface of P' . Then we consider the following function on $h_1 : P \rightarrow \mathbf{R}$. For every point $p \in P$ we draw a line l from n to p and define $h(p)$ as the ratio of the distances from n to p and from n to the outer boundary of P' . This function will be piecewise linear on the triangulation of P that is obtained by intersecting P with $\text{Conv}(n, F)$ for all faces F of the outer boundary of P' . Moreover, it will be strictly convex in a sense that for each two $p_1, p_2 \in P$ and each $\alpha \in (0, 1)$

$$\alpha h(p_1) + (1 - \alpha)h(p_2) \leq h(\alpha p_1 + (1 - \alpha)p_2)$$

with equality satisfied if and only if there exists a simplex of the triangulation that contains both p_1 and p_2 . We then extend h from P to C^* by putting $h_{C^*}(p) = h(p)(\deg \cdot p)$. This function will be strictly convex on the triangulation of C^* such that all its simplices contain n , which finishes the proof. \square

Remark 9.9 Looking back, we really had to work hard to prove last two propositions. It would be very interesting to find a direct proof not based on results of Proposition 6.6.

Proposition 9.10 If C_1^* is a face of C^* then surjective map $\text{Fock}_{M \oplus C^*} \rightarrow \text{Fock}_{M \oplus C_1^*}$ induces a map $\mathcal{V}_C \rightarrow \mathcal{V}_{C_1}$ which is a localization map of loop-module \mathcal{V}_C over $\mathbf{C}[C]$ by multiplicative system $S = \{x^m, m \cdot C_1^* = 0, m \cdot C^* \geq 0\}$.

Proof. The hard part was to show that spaces in question are loop-modules, that is to show that $L[0]$ is non-negative. Then argument of Proposition 7.1 extends to this more general situation. \square

For the rest of this section we no longer assume that $\deg \in M$ is fixed.

Proposition 9.11 A toric variety \mathbf{P} defined by fan Σ is Gorenstein if for every cone $C^* \in \Sigma$ all generators n of one-dimensional cones of C satisfy $\deg_C \cdot n = 1$ where \deg_C is a lattice point in M .

Proof. See [1] \square

The following definition is made possible by the results of Propositions 9.10 and 9.7.

Definition 9.12 Let \mathbf{P} be a Gorenstein toric variety, given by fan Σ in N . Fix a generic set of numbers g_n for all points of degree one in each cone of Σ (the notion of degree may vary from cone to cone). Then (generalized) chiral de Rham complex $\mathcal{MSV}(\mathbf{P})$ is defined as a quasi-loco sheaf over it such that for any affine subspace \mathbf{A}_C of \mathbf{P} sections of $\mathcal{MSV}(\mathbf{P})$ are BRST cohomology of $\text{Fock}_{M \oplus C^*}$ by operator

$$\mathcal{BRST}_g = \oint \left(\sum_{n \in C^*, \deg_C \cdot n = 1} g_n(n \cdot \Psi)(z) e^{\int n \cdot A(z)} \right) dz.$$

Remark 9.13 Notice that while the choice of g is irrelevant in the smooth or even orbifold case, it is very important in general.

Theorem 9.14 Quasi-loco sheaf $\mathcal{MSV}(\mathbf{P})$ is in fact a loco sheaf. Recall that the grading is given by $L[0]$ where

$$L(z) = : B(z) \cdot A(z) : + : \partial_z \Phi(z) \cdot \Psi(z) : .$$

Proof. This is essentially a local statement, so it is enough to work with one cone C^* of maximum dimension. We need a few preliminary lemmas.

Lemma 9.15 The action of $\deg \cdot B[0]$ could be pushed down to the BRST cohomology of $\text{Fock}_{M \oplus K^*}$. BRST cohomology of $\text{Fock}_{M \oplus K^*}$ has only eigenvalues of $\deg \cdot B[0]$ in a certain range (from 0 to D_1).

Proof of the Lemma. Our BRST operator increases $\deg \cdot B[0]$ eigen-values by one, so the action of $\deg \cdot B[0]$ could be pushed down to the cohomology. As before, we can notice that the above statement is true for any simplicial subcone and then do the spectral sequence and degeneration trick as in Proposition 9.7. We must mention that as a result of the spectral sequence the bound D_1 could jump a bit, but we only need to know that such a bound exists. \square

We will need some general theory of funny objects which we call *almost-modules*.

Definition 9.16 Let R be a Noetherian ring. An abelian group V is called an *almost-module* over R if

- there is defined a map $R \times V \rightarrow V$ which is bilinear but not necessarily associative;
- there exists a finite filtration of V compatible with the multiplication map above such that the quotients of the filtration are modules over R .

Lemma 9.17 Let V be an almost-module over R . The following conditions are equivalent:

- (1) There exists a filtration of V as above such that all quotients are finitely generated.
- (2) For any filtration of V as above all quotients are finitely generated.
- (3) Any ascending chain of sub-almost-modules terminates.

If these conditions hold, then V is called a Noetherian almost-module.

Proof of the lemma. (3) \Rightarrow (2). If one of the quotients is not finitely generated, then there will be an ascending non-terminating chain of submodules inside it. It will give rise to an ascending non-terminating chain of sub-almost-modules of V . (1) \Rightarrow (3). If

$$V = F^k V \subset F^{k-1} V \subset \dots \subset F^0 V = 0$$

is the filtration on V then for any ascending chain $\{V_j\}$ one can look at $V_j \cap F^1 V$. At some point it will stabilize. Then we can look at $V_j \cup F^2 V / F^1 V$, and it will stabilize too. This will imply that $V_j \cup F^2 V$ is stabilized. Eventually we will have $V_j \cup F^k V = V_j$ stabilized. \square

As a corollary from the above proposition, all submodules and quotients of Noetherian almost-modules are Noetherian.

Lemma 9.18 For any d, D and any finite set $I \subseteq N$ the $L[0] = d$ part of \mathcal{BRST}_g cohomology of $\text{Fock}_{K-D \deg \oplus I}$ is a Noetherian almost-module over $\mathbf{C}[K]$. The multiplication is given by $e^{\int m \cdot B}[0]$.

Proof of the lemma. Consider the filtration by the number of A . When $e^{\int B \cdot m}[0]$ acts on the quotients of this filtration, it amounts to acting on $|m_1, n_1\rangle$ directly, because one can push through and the extra commutators are in the lower part of the filtration. We can also do it for one fixed n , because there are only finitely many of them.

Then we see that the multiplication gives zero unless $m \cdot n = 0$, and for those m it is simply a shift of $A[0]$ eigen-values. Now it remains to notice that there are finitely many (linearly independent) choices of extra A, B, Φ, Ψ to get the desired $L[0]$ eigen-value. Also, there are finitely many choices for $m \cdot n$ in $m, n\rangle$ because it can't be too big. And for each $m \cdot n = \text{const}$ we have a finitely generated module over $\mathbf{C}[K]$ which finishes the proof of the lemma. \square

We are now ready to complete the proof of Theorem 9.14. What we need to show is that $L[0] = d$ component of \mathcal{BRST}_g cohomology of $\text{Fock}_{M \oplus K^*}$ is a Noetherian almost module over $\mathbf{C}[K]$. Because of Proposition 9.8 and Lemma 9.15, it is enough

to consider $\text{Fock}_{K-D \deg, \{\deg \cdot \leq D_1\}}$. By the above lemma, this is a Noetherian almost-module, and the observation after Lemma 9.17 finishes the proof of the theorem. \square

The above theorem together with Proposition 2.9 lead to the following corollary.

Corollary 9.19 If \mathbf{P} is compact, then $H^*(\mathcal{MSV}(\mathbf{P}))$ is a graded vertex algebra with finite dimensional graded components.

It is worthwhile to mention that $\mathcal{MSV}(\mathbf{P})$ is also loop-coherent with respect to the grading by the B model Virasoro operator $L_B[0]$ which is the zeroth mode of

$$L_B(z) = :B(z) \cdot A(z): + : \partial_z \Psi(z) \cdot \Phi(z): - \deg \cdot B(z).$$

Theorem 9.20 $\mathcal{MSV}(\mathbf{P})$ is loop-coherent with respect to the grading by $L_B[0]$.

Proof. The proof is completely analogous to the proof of Theorem 9.14. We simply follow the chain of propositions of this section, with the following change. In the proof of Proposition 9.5, in addition to the table

mode	$a_i[-k]$	$b^i[-k]$	$\varphi^i[-k]$	$\psi_i[-k]$
$\Delta L_B[0]$	$k - \alpha_i$	$k + \alpha_i$	$k + 1 + \alpha_i$	$k - 1 - \alpha_i$

we need to consider the extra shift by $\deg \cdot \alpha = \sum_i \alpha_i$ due to the extra term in $L_B(z)$. This shift cures possible negative contributions of $\psi_i[-1]$ (fortunately, these are fermionic modes so they can not repeat). \square

We will now address the problem of *string-theoretic cohomology vector spaces*. Recall that string cohomology numbers were constructed by Batyrev and Dais in [5], and it was proved in [4] that they comply with predictions of Mirror Symmetry for Calabi-Yau complete intersections in toric varieties. Unfortunately, until now it was not known how to construct string cohomology vector spaces whose dimensions are the string cohomology numbers above. The analysis of this paper suggests the following definition, at least for toric varieties.

Definition 9.21 *String-differential forms* on \mathbf{P} is $L[0] = 0$ part of $\mathcal{MSV}(\mathbf{P})$. By Remark 2.8 it is a coherent sheaf.

The following proposition provides us with a much more practical definition of this sheaf that does not refer explicitly to sheaves of vertex algebras.

Proposition 9.22 For each $C^* \in \Sigma$ consider $\mathbf{C}[C \oplus C^*]$ -module V_C defined as

$$V_C = \bigoplus_{m \in C, n \in C^*, m \cdot n = 0} \mathbf{C} x^m y^n$$

where the action of x^k and y^l is defined to be zero if the result violates $m \cdot n = 0$. There is defined a differential $\mathcal{BRST}_g^{(0)}$ on $V_C \otimes_{\mathbf{C}} (\Lambda^* M_{\mathbf{C}})$ given by

$$\mathcal{BRST}_g^{(0)} = \sum_{n \in C^* \cap \Delta^*} g_n y^n \text{contr}(n)$$

where $\text{contr}(n)$ indicates contraction by n on Λ^*M . Then cohomology of $\mathcal{BRST}_g^{(0)}$ is a finitely generated $\mathbf{C}[C]$ -module isomorphic to $L[0] = 0$ component of \mathcal{V}_C . Moreover, grading by $J[0]$ on it is defined as "degree in Λ^*M plus degree of n ", and differential d is defined as

$$d(wx^m y^n) = (w \wedge m)x^m y^n.$$

Proof. Due to Proposition 9.8, $L[0] = 0$ part of \mathcal{V}_C could be obtained by applying \mathcal{BRST}_g to $L[0] = 0$ part of $\text{Fock}_{C \oplus C^*}$. For every $|m, n\rangle$ from this space we already have $m \cdot n = 0$, so all elements from this space are obtained by multiplying $|m, n\rangle$ by products of $\Phi^i[0]$, that is by Λ^*M . Then we only need to calculate the action of \mathcal{BRST}_g on this space, as well as the actions of $J[0]$ and $Q[0]$. This is accomplished by a direct calculation. \square

If desired, one can use the above proposition as a definition of the space of sections of the sheaf of string-differential forms over \mathbf{P} . It is not hard to show that it is coherent directly.

Remark 9.23 Notice that sheaf of string-differential forms is not locally free, it reflects singularities of \mathbf{P} .

Remark 9.24 Another peculiar feature of the above description is that grading by eigen-values of $J[0]$ on the space of differential forms seems to be ill-defined, since $J[0]$ varies with the cone. Nevertheless, this is not a problem, because the notion of $J[0]$ behaves well under the localization, so string cohomology spaces do have an expected double grading. Besides, de Rham operator $\oint Q(z)dz$ is clearly well-defined for string-differential forms on \mathbf{P} .

Remark 9.25 The fiber of the sheaf of string-differential forms over the most singular point of \mathbf{A}_C is obtained by considering only $m = 0$ part of the above space. It is easily seen to coincide with the prediction of [6].

Remark 9.26 It is not entirely clear if one should consider the cohomology of $\mathcal{MSV}(\mathbf{P})$ or the hypercohomology of it under the $Q(0)$ operator. On one hand, hypercohomology might be a smaller and nicer object, but on the other hand taking hypercohomology may complicate the relation between A and B models. So our definitions below should be considered only provisional.

Definition 9.27 String cohomology vector space is the hypercohomology of the complex of string-differential forms.

It is very likely that our definition reproduces correctly the numbers of [5], but clearly more work is necessary. We would like to formulate this as a vague conjecture.

Conjecture 9.28 For every variety X with only Gorenstein toroidal singularities there exists a loco sheaf $\mathcal{MSV}(X)$ which is locally isomorphic to the product of $\mathcal{MSV}(\text{open ball})$ and $\mathcal{MSV}(\text{singularity})$ defined above. This construction depends on the choice of parameters g_n and perhaps on some other structures yet to be determined. Sheaf $\mathcal{MSV}(X)$ is provided with the structure of a sheaf of conformal vertex algebras, and with $N = 2$ structure if X is Calabi-Yau. The $L[0] = 0$ component of this sheaf has a natural grading and differential which generalizes de Rham differential. The hypercohomology of this complex has dimensions prescribed by [5] and possesses a pure Hodge structure if X is projective.

10 Hypersurfaces in Gorenstein Toric Fano Varieties: general case

Even though we are unable to construct chiral de Rham complex for an arbitrary variety with Gorenstein toroidal singularities, the situation is somewhat better in the special case of a hypersurface X in a Gorenstein toric variety. We can use the formulas for the smooth case applied now to arbitrary cones. The resulting sheaf turns out to be loop-coherent. We are mostly interested in the case when the ambient variety \mathbf{P} is Fano and the hypersurface X is Calabi-Yau and generic, but most statements hold true for any generic hypersurfaces. We will try to extend the calculation of Sections 6-8.

We use the same notations Δ , Δ^* , Σ , $M = M_1 \oplus \mathbf{Z}$, $N = N_1 \oplus \mathbf{Z}$, \deg , \deg^* as in Sections 7 and 8. We again consider a projective variety \mathbf{P} , a line bundle L on it, and a hypersurface X in \mathbf{P} given by $f : \Delta \rightarrow \mathbf{C}$. We define $\mathcal{MSV}(X)$ as follows.

Definition 10.1 Let $f : \Delta \rightarrow \mathbf{C}$ be a set of coefficients that defines X and $g : \Delta^* \rightarrow \mathbf{C}$ be a generic set of parameters. Then for any cone $C^* \in \Sigma$ that contains \deg^* sections of quasi-loco sheaf $\mathcal{MSV}(X)$ over the affine chart \mathbf{A}_C are defined as BRST cohomology of $\text{Fock}_{M \oplus C^*}$ with BRST operator

$$\mathcal{BRST}_{f,g} = \oint \left(\sum_{m \in \Delta} f_m(m \cdot \Phi)(z) e^{\int m \cdot B(z)} + \sum_{n \in \Delta^*} g_n(n \cdot \Psi)(z) e^{\int n \cdot A(z)} \right) dz.$$

For the above definition to make sense, we should show that the spaces of sections constructed above are compatible with localization. Moreover, we must show that they are loop-modules over the structure ring of X , which means that they are annihilated by f .

Proposition 10.2 The above definition indeed defines a quasi-loco sheaf of vertex algebras over X . It is provided with the structure of topological algebra by formulas of Proposition 7.15.

Proof. To prove compatibility with localizations, we need to show that for any cone C^* of maximum dimension the $\mathcal{BRST}_{f,g}$ cohomology of $\text{Fock}_{M \oplus C^*}$ is non-negatively graded with respect to $L[0]$. Then the argument of Proposition 7.1 shows the compatibility. The field $L[z]$ here is given by the formulas of Proposition 7.15, in particular, it differs slightly from $L(z)$ of the Section 9. To avoid confusion we will call the operator given in 7.15 by $L_{X,A}[0]$. This notation is chosen to indicate that we are dealing with the Virasoro algebra of A model on the hypersurface X . Explicitly,

$$L_{X,A}(z) = :B(z) \cdot A(z): + : \partial_z \Phi(z) \cdot \Psi(z): - \deg^* \cdot \partial_z A(z)$$

so $L_{X,A}[0]$ counts the opposite of the sum of mode numbers of A, B, Φ, Ψ plus $m \cdot n$ plus $m \cdot \deg^*$. One can split $\mathcal{BRST}_{f,g}$ as a sum of \mathcal{BRST}_f and \mathcal{BRST}_g as usual. Then we will have a spectral sequence as in Proposition 7.11. It is easily shown to be convergent because of Lemma 9.15. As a result, it is enough to show that the \mathcal{BRST}_g cohomology of $\text{Fock}_{M \oplus C^*}$ has nonnegative $L_{X,A}[0]$ eigenvalues.

Since C^* contains \deg^* and \mathbf{P} is Gorenstein, cone C has some special properties. One of the generators of its one-dimensional faces is a vertex m_0 of Δ , and all other generators lie in M_1 . Notice that $\text{Fock}_{M \oplus C^*}$ naturally splits as a tensor product of $\text{Fock}_{M_1 \oplus C_1^*}$ and $\text{Fock}_{\mathbf{Z}m_0 \oplus \mathbf{Z}_{\geq 0} \deg^*}$. Here C_1^* is the cone in N_1 obtained by projecting C^* there along \deg^* . Moreover, it is easy to see that the \mathcal{BRST}_g cohomology of $\text{Fock}_{M \oplus C^*}$ is the tensor product of \mathcal{BRST}_g cohomology of $\text{Fock}_{M_1 \oplus C_1^*}$ and \mathcal{BRST}_g cohomology of $\text{Fock}_{\mathbf{Z}m_0 \oplus \mathbf{Z}_{\geq 0} \deg^*}$. BRST operator on the first space is defined precisely as in the Section 9, and BRST operator on the second space is

$$\mathcal{BRST}(z) = \deg^* \cdot \Psi(z) e^{\int \deg^* \cdot A(z)}.$$

Moreover, $L_{X,A}[0]$ is the sum of $L[0]$ from the Section 9 applied to $M_1 \oplus C_1^*$ and

$$L_2[0] = : (m_0 \cdot B)(\deg^* \cdot A): [0] + : (\partial m_0 \cdot \Phi)(\deg^* \cdot \Psi): [0] + \deg^* \cdot A[0]$$

Proposition 9.7 assures that BRST cohomology of $M_1 \oplus C_1^*$ doesn't have negative eigenvalues of $L_{X,A}[0]$. Explicit calculation of BRST cohomology of $\text{Fock}_{\mathbf{Z}m_0 \oplus \mathbf{Z}_{\geq 0} \deg^*}$ given in Theorem 6.5 allows us to conclude that $L_2[0]$ also has non-negative eigenvalues. This assures that $L_{X,A}[0]$ has only non-negative eigenvalues.

However, we have only shown so far that $\mathcal{MSV}(X)$ is a sheaf of vertex algebras over \mathbf{P} . We also need to prove that the structure of $\mathbf{C}[C]$ -loop-module induced from $\text{Fock}_{M \oplus C^*}$ naturally gives the structure of $\mathbf{C}[C \cap M_1]/r_f$ -loop-module, where r_f is the local equation of the hypersurface. It is enough to show this for a cone C^* of maximum dimension. Locally the element r_f is

$$\sum_{m \in \Delta} f_m x^{m-m_0}.$$

Notice that the corresponding field

$$r_f(z) = \sum_{m \in \Delta} f_m e^{(m-m_0) \cdot B(z)}$$

in $\text{Fock}_{M \oplus C^*}$ could be expressed as an anti-commutator of $\mathcal{BRST}_{f,g}$ and

$$R(z) = \deg^* \cdot \Psi(z) e^{\int -m_0 \cdot B(z)}.$$

Really, the OPE of this field with $\mathcal{BRST}_g(w)$ is non-singular, because either $n \neq \deg^*$ and $-m_0 \cdot n = 0$, or $n = \deg^*$, which gives $-m_0 \cdot n = -1$. However, in the latter case, we will also have $\deg^* \Psi(z) \deg^* \Psi(w)$ which is $O(z-w)$, so overall the OPE is still non-singular. Hence, all modes of $r_f(z)$ act trivially on the cohomology by $\mathcal{BRST}_{f,g}$, which finishes the proof of the proposition.

We remark that it is plausible that all modes of $e^{m_0 \cdot B(z)}$ act trivially as well, but we do not need to prove it, because $\mathbf{C}[C \cap M_1]$ is embedded in $\mathbf{C}[C]$. \square

It seems certain that $\mathcal{MSV}(X)$ is loco with respect to $L_{X,A}[0]$, but we do not have a proof of it yet. It would follow from any reasonable solution of Conjecture 9.28. On the other hand, we can easily show that $\mathcal{MSV}(X)$ is loco with respect to the grading $L_{X,B}[0]$ that comes from the B model Virasoro field. This field is given by

$$L_{X,B}(z) = :B(z) \cdot A(z) : - : \Phi(z) \cdot \partial \Psi(z) : -\deg \cdot \partial_z B(z).$$

Theorem 10.3 $\mathcal{MSV}(X)$ is a loop-coherent sheaf with respect to the grading $L_{X,B}[0]$.

Proof. The question is local and it is enough to consider a cone C^* of maximum dimension. We are working in the set-up of the previous proposition. By Theorem 9.20, cohomology of $\text{Fock}_{M \oplus C^*}$ with respect to \mathcal{BRST}_g has graded components that are Noetherian almost-modules over $\mathbf{C}[C]$. Really, expressions for L_B and $L_{X,B}$ are identical (which is not the case for A model).

Notice that for a sufficiently big integer k , we can express

$$e^{k m_0 \cdot B(z)}$$

as an anticommutator of some field and $\mathcal{BRST}_{f,g}$. That field is similar to the one in the proof of Proposition 8.1, but without the extra $e^{-k m_0 \cdot B}$. The spectral sequence from \mathcal{BRST}_f cohomology of \mathcal{BRST}_g cohomology to the $\mathcal{BRST}_{f,g}$ cohomology degenerates by Lemma 9.15, so $\mathcal{BRST}_{f,g}$ cohomology is has graded components that are Noetherian almost-modules over $\mathbf{C}[C]/(x^{k m_0})$. Since we have already shown that r_f acts trivially, these spaces are Noetherian over the structure ring of X . \square

Corollary 10.4 For a fixed pair of eigen-values of $L[0]$ and $J[0]$, the corresponding eigen-spaces of $H^*(\mathcal{MSV}(X))$ are finite-dimensional.

It is our firm belief that after Conjecture 9.28 is successfully proved, the sheaf $\mathcal{MSV}(X)$ could be identified as a (generalized) chiral de Rham complex of X as implied by this notation. However, it is still well defined as a sheaf of vertex algebras and one may ask how to calculate its cohomology.

We can also ask whether the analog of Proposition 7.11 still holds. For orbifold singularities Proposition 7.11 still holds the way it is stated, but the proof must

be different, because vertical cohomology of the double complex considered there is non-zero for more than one row. In the case of orbifold singularities spectral sequence still degenerates, because we may split the picture according to the eigenvalues of $B[0]$ modulo the lattice spanned by generators of one-dimensional faces of C^* . This spectral sequence might degenerate for every C^* , but we can not prove it.

Unfortunately, Proposition 7.12 does not hold even for orbifold singularities. It is also not clear that double complex of Theorem 7.14 gives degenerate spectral sequences for arbitrary Gorenstein toric Fano varieties. On the other hand, $\mathcal{BRST}_{f,g}$ cohomology of $\text{Fock}_{M \oplus K^*}$ could still be the correct vertex algebra to consider, once the relation to physicists' A and B models becomes more clear. Section 8 never uses the fact that \mathbf{P} is non-singular and generalizes to any \mathbf{P} . To sum it up, it is plausible that Theorem 8.3 holds for any toric Gorenstein Fano varieties, but we can only prove it in the smooth case.

To complete the discussion we must mention that Mirror Symmetry for Calabi-Yau complete intersections in Gorenstein toric Fano varieties can be adequately treated by the methods of this paper. It is appropriate to state the final conjecture that covers all examples of "toric" Mirror Symmetry. See [3] for relation between complete intersection examples of Mirror Symmetry and pairs of dual reflexive Gorenstein cones.

Conjecture 10.5 Let M and N be dual lattices with dual cones K and K^* in them. We assume that K and K^* are reflexive Gorenstein, which means that $K \oplus K^*$ is Gorenstein in $M \oplus N$. We denote the corresponding degree elements by \deg and \deg^* . We are also provided with generic numbers g_n and f_m for all elements in K and K^* . Then, if reflexive cones come from Calabi-Yau complete intersections, vertex algebras of these Calabi-Yau manifolds are degenerations of BRST cohomology of $\text{Fock}_{M \oplus N}$ by operator

$$\mathcal{BRST}_{f,g} = \oint \left(\sum_m f_m (m \cdot \Phi)(z) e^{\int m \cdot B(z)} + \sum_n g_n (n \cdot \Psi)(z) e^{\int n \cdot A(z)} \right) dz.$$

The degeneration is provided by fans that define the corresponding toric varieties. Structure of topological algebras of dimension $\dim M - 2\deg \cdot \deg^*$ is given by

$$Q(z) = A(z) \cdot \Phi(z) - \deg \cdot \partial_z \Phi(z), \quad G(z) = B(z) \cdot \Psi(z) - \deg^* \cdot \partial_z \Psi(z),$$

$$J(z) = : \Phi(z) \cdot \Psi(z) : + \deg \cdot B(z) - \deg^* \cdot A(z),$$

$$L(z) = : B(z) \cdot A(z) : + : \partial_z \Phi(z) \cdot \Psi(z) : - \deg^* \cdot \partial_z A(z).$$

11 Open questions and concluding remarks

In this section we point out important questions that were not addressed in this paper as well as possible applications of our results and techniques.

- It remains to show that deformations of the Master Family of vertex algebras are flat in the appropriate sense. For instance, we would love to say that dimensions of $L[0]$ eigen-spaces are preserved under these deformations.
- One can generalize the construction of Conjecture 10.5 to go from $K \oplus K^*$ in $M \oplus N$ to any Gorenstein self-dual cone in a lattice with inner product by using the BRST field

$$\mathcal{BRST}(z) = \sum_n g_n(n \cdot \text{fermion})(z) e^{\int n \cdot \text{boson}(Z)}.$$

These theories still have conformal structure, with L given as $L_{\text{flat}} - \deg \cdot \partial \text{boson}$. Do these theories have any nice properties or physical significance?

- It would be great to provide vertex algebras of Mirror Symmetry with unitary structure. Inequalities on eigen-values of $L[0]$ and $J[0]$ seem to suggest its existence.
- It is extremely important to use results of this paper to get actual correlators of corresponding conformal field theories and thus to draw a connection to the calculation of [7].
- There must be a connection between calculations of this paper and moduli spaces of stable maps defined by Kontsevich. It remains a mystery at this time.
- It would be interesting to see how GKZ hypergeometric system enters into our picture. Solutions of GKZ system are known to give cohomology of Calabi-Yau hypersurfaces, see for example [12, 21].
- One must also do something about the antiholomorphic part of the $N = (2, 2)$ super-conformal algebra. Perhaps this problem is not too hard and has its roots in the author's ignorance.
- One should construct generalized chiral de Rham complexes as suggested in Conjecture 9.28. This seems to be a realistic project since all we really need to do is to extend the automorphisms of toroidal singularities to the suggested local descriptions of chiral de Rham complexes. It is also interesting to see which of the standard properties of cohomology of smooth varieties generalize to string-theoretic cohomology.
- One should define string cohomology for all Gorenstein, and perhaps \mathbf{Q} -Gorenstein singularities. See [2] for the definition of string cohomology numbers in this generality. This may also shed some light on generalized McKay correspondence.
- Our results and techniques may have applications to hyperbolicity. Indeed, for a smooth X there is a loco subsheaf $\mathcal{MSV}_{b,\varphi}$ of $\mathcal{MSV}(X)$ which is generated by modes of b^i and φ^i only. This part of $\mathcal{MSV}(X)$ is contravariant, so for any map from a line to X it could be pulled back to it. Then global sections of $\mathcal{MSV}_{b,\varphi}$ could give restrictions on possible maps to X . On the other hand, rich structure of the whole $\mathcal{MSV}(X)$ might help to show that there are plenty of such sections.
- Finally, cohomology of $\mathcal{MSV}(X)$ is graded by $L[0]$ and $J[0]$, and one can show that $\text{Trace}(q^{L[0]} w^{J[0]})$ has some modular properties. It is directly related to elliptic genus of X . This issue will be addressed in the upcoming joint paper with Anatoly Libgober.

References

- [1] V. V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, J. Algebraic Geom., **3** (1994) 493-535.
- [2] V. V. Batyrev, *Stringy Hodge numbers and Virasoro algebra*, preprint alg-geom/9711019.
- [3] V. V. Batyrev, L. A. Borisov, *Dual Cones and Mirror Symmetry for Generalized Calabi-Yau Manifolds*, Mirror Symmetry II (B. Greene and S.-T. Yau, eds.), International Press, Cambridge, 1997, 65-80 (1995).
- [4] V. V. Batyrev, L. A. Borisov, *Mirror Duality and String-theoretic Hodge Numbers*, Invent. Math., **126**(1996), Fasc. 1, 183-203.
- [5] V. V. Batyrev, D. I. Dais, *Strong McKay Correspondence, String-theoretic Hodge Numbers and Mirror Symmetry*, Topology, **35**(1996), 901-929.
- [6] L. A. Borisov, *String Cohomology of a Toroidal Singularity*, preprint alg-geom/9802052.
- [7] P. Candelas, X. C. de la Ossa, P. S. Green, and L. Parkes, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, Nuclear Phys. B **359** (1991), 21-74.
- [8] V. I. Danilov, *The Geometry of Toric Varieties*, Russian Math. Surveys, **33**(1978), 97-154.
- [9] W. Fulton, *Introduction to toric varieties*, Princeton University Press (1993).
- [10] A. B. Givental, *Equivariant Gromov-Witten Invariants*, preprint alg-geom/9603021.
- [11] B. R. Greene, *String Theory on Calabi-Yau Manifolds*, preprint hep-th/9702155.
- [12] S. Hosono, *GKZ Systems, Gröbner Fans and Moduli Spaces of Calabi-Yau Hypersurfaces*, preprint alg-geom/9707003.
- [13] V. Kac, *Vertex algebras for beginners*, University Lecture Series, **10**, American Mathematical Society, Providence, RI, 1997.
- [14] M. Kontsevich, *Enumeration of rational curves via torus actions*, preprint hep-th/9405035.
- [15] W. Lerche, C. Vafa, P. Warner, *Chiral rings in $N=2$ superconformal theories*, Nucl. Phys. **B324** (1989), 427-474.
- [16] B. Lian, K. Liu, S.T. Yau, *Mirror Principle I*, preprint alg-geom/9712011.
- [17] F. Malikov, V. Schechtman, A. Vaintrob, *Chiral de Rham complex*, preprint alg-geom/9803041.

- [18] D. R. Morrison, *Making Enumerative Predictions by Means of Mirror Symmetry*, Mirror Symmetry II (B. Greene and S.-T. Yau, eds.), International Press, Cambridge, 1997, 457-482.
- [19] T. Oda, *Convex Bodies and Algebraic Geometry - An Introduction to the Theory of Toric Varieties*, Ergeb. Math. Grenzgeb. (3), vol. 15, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1988.
- [20] A. Schwarz, *Sigma-models having supermanifolds as target spaces*, Lett. Math. Phys. **38** (1996), 91.
- [21] J. Stienstra, *Resonant Hypergeometric Systems and Mirror Symmetry*, preprint alg-geom/9711002.
- [22] E. Witten, *Mirror manifolds and topological field theory*, Essays on Mirror Manifolds (S.-T. Yau, ed.), International Press, Hong Kong, 1992, 120–159.